The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before August 1, 2009. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Fall, 2009 issue of *The Pentagon*. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu, fax: (859)-622-3051)

NEW PROBLEMS 632-640

Problem 632. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, IL.*

Let two rays meet at point A, and let P be a point on one ray and Q a point on the other ray. Let B be a point between A and P. Suppose the angle measure of $\angle PAQ$ is less than 60°. Show how to construct, with only compass and straightedge, points D on AP and C on AQ such that $CD = AB$ and DC makes an angle of 60° with AQ.

Problem 633. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, IL.*

The integers beginning with 2008 and without spaces between them are written down:

$$200820092010201120122013...$$

Then commas are placed to form an infinite sequence of 5-digit arrangements:

$$20082, 00920, 10201, 12012, 20132, \ldots$$

Prove or disprove: Every 5-digit arrangement appears infinitely many times in this sequence.

Problem 634. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.*

Find the limit

$$\lim_{n\to\infty} \frac{1}{n} \left(\frac{n}{\frac{1}{2} + \frac{2}{3} + \cdots + \frac{n}{n+1}} \right)^n .$$

Problem 635. *Proposed by Ovidiu Furdui, University of Toledo, Toledo, OH.*

Let $k \geq 2$ be a natural number. Find the sum

$$\sum_{n_1, \ldots, n_k \geq 1} \left(\zeta \left(n_1 + \cdots + n_k \right) - 1 \right),$$

where ζ denotes the Riemann zeta function.

Problem 636. *Proposed by Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO.*

Let p be a fixed prime. Find the dimensions of all rectangles with integral side lengths and whose areas are numerically equal to p times their semiperimeters.

Problem 637. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let $x, y, z \in [1, \infty)$. Prove that

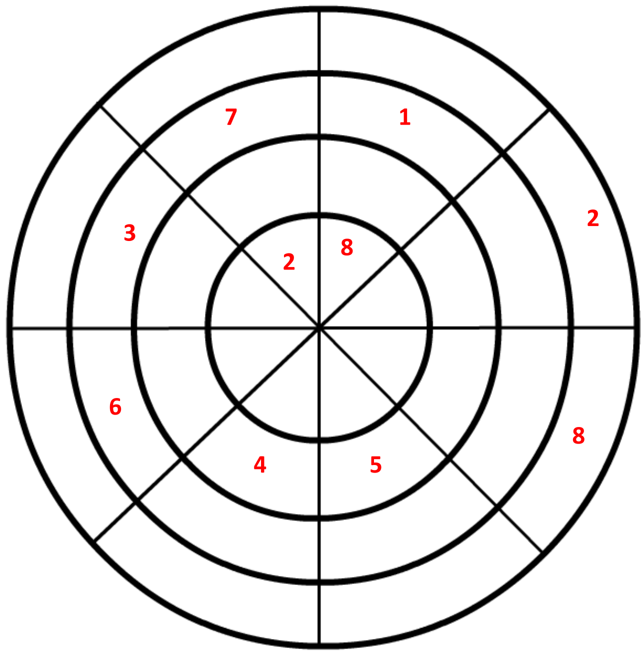
$$\frac{x}{x^2 + yz} + \frac{y}{y^2 + xz} + \frac{z}{z^2 + xy} \leq \frac{3}{2}.$$

Problem 638. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let a be a positive integer. Find the least common multiple of the number $A = a^n (a+1)^{n+1} + a$ and $B = a^{n+1} (a+1)^n + a - 1$, where n is any natural number.

Problem 639. *Proposed by Peter M. Higgins and Caroline Higgins (authors of the book Circular Sudoku), Essex University, England.*

The following is a Circular Sudoku puzzle. Each of the numbers 1-8 must appear once in every ring and once in every pair of touching slices. Fill in the missing values of the puzzle.



Problem 640. *Proposed by the editor*.

The sequence 19,199,1999,... starts off with three primes; most of the numbers in the sequence, however, are composites, and there are lots of divisors of the numbers in the sequence. Prove the following:

1. The prime 19 divides infinitely many of the numbers in the sequence.
2. The composite number 551 divides infinitely many of the numbers in the sequence.
3. The composite number 323 does not divide any of the numbers in the sequence.

SOLUTIONS 616-623

Problem 616. *Proposed by Melissa Erdmann, Nebraska Wesleyan University, Lincoln, NE.*

The birthday paradox is that in a room with 23 people, the probability that two or more of them will have the same birthday (month and day) is at least 50%. Find the number of people needed so that there is a 50% probability that at least three or more of them will have the same birthday. Find the formula that will represent the probability that at least k of n total people in a room share a birthday.

Solution *by the proposer*.

In order to compute the probabilities of at least 3 people sharing a birthday in a room of n people, we compute the probability of no people sharing a birthday and the probabilities of 1 pair having the same birthday, 2 pairs having the same birthday,...,$[n/2]$ pairs having the same birthday, and subtracting all of these probabilities from 1. Using 365 days as the number of possible birthdays, the probability of no two people sharing the same birthday is

$$\frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+1}{365}.$$

The probability of one pair of people sharing the same birthday is

$$\binom{n}{2} \cdot \frac{365}{365} \cdot \frac{1}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{365-n+2}{365},$$

where the binomial coefficient gives the number of ways to choose the pair of people, the 365 in the numerator is the number of days for this pair to have in common, the 1 after it represents that the second person in the pair has only the one choice for birthday, and then the remaining people all have distinct birthdays.

Similarly, the probability of two pair of people sharing the same birthday is

$$\binom{n}{2} \cdot \frac{365}{365} \cdot \frac{1}{365} \cdot \frac{\binom{n-2}{2}}{2!} \cdot \frac{364}{365} \cdot \frac{1}{365} \cdot \frac{363}{365} \cdots \frac{365-n+3}{365},$$

where the second binomial coefficient reflects that there are two fewer people to select from for the second pair, the 2! below it represents that the two selected pairs of people could have been chosen in the other order, and the 364 in the numerator following this represents that the second selected pair has a common birthday different from the first pair. We continue this process for $k = 2, ..., [n/2]$ pairs of people having the same birthdays. We can program this in Mathematica in the following manner:

$$N\left[1-\left(\sum_{k=0}^{\text{Floor}[n/2]}(365!/(365-(n-k))!\right.\right.$$
$$\left.\left.\cdot\left(\prod_{i=1}^{k}\text{Binomial}[n-2(i-1),2]\right)/k!)\right)/(365^n)\right]$$

When executed with $n = 87$, we get a probability of 0.499455. When executed with $n = 88$, we get a probability of 0.511065. Hence 88 is the smallest number of people needed in a room so that the probability is greater than 50% that 3 people share the same birthday.

The formula for the smallest number of people needed in a room so that the probability is greater than 50% that k people share the same birthday gets much more complicated because of the number of cases to handle. It turns out that 187 people are needed so that 4 people share the same birthday with a probability greater than 0.5. Several additional values are given at the following website in a 1998 Ivars Peterson column for Science News entitled "Birthday Surprises." `http://www.sciencenews.org/sn_arc98/11_21_98/mathland.htm`.

Problem 617. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Find all triplets (x, y, z) of real numbers such that

$$\sqrt{3^x\left(4^y+5^z\right)}+\sqrt{4^y\left(3^x+5^z\right)}+\sqrt{5^z\left(3^x+4^y\right)}=\sqrt{2}\left(3^x+4^y+5^z\right).$$

Solution *by the proposer*.

Setting $a = 3^x$, $b = 4^y$, and $c = 5^z$, the equation given above becomes

$$\sqrt{a\,(b+c)}+\sqrt{b\,(a+c)}+\sqrt{c\,(a+b)}=\sqrt{2}\,(a+b+c)\,.$$

Using the AM-GM inequality yields

$$\sqrt{a\,(b+c)}\leq\sqrt{2}\left(\frac{a}{2}+\frac{b+c}{4}\right),$$

$$\sqrt{b\,(a+c)}\leq\sqrt{2}\left(\frac{b}{2}+\frac{a+c}{4}\right),$$

and

$$\sqrt{c\,(a+b)}\leq\sqrt{2}\left(\frac{c}{2}+\frac{a+b}{4}\right).$$

Adding these inequalities, we obtain

$$\sqrt{a\,(b+c)}+\sqrt{b\,(a+c)}+\sqrt{c\,(a+b)}=\sqrt{2}\,(a+b+c)\,.$$

Equality holds when $a=b=c$. Therefore, the solutions to the equation of the problem are when $3^x=4^y=5^z$; that is, when $x=y=z=0$.

Problem 618. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let a, b, c be real numbers such that $0<a\le b\le c<\dfrac{\pi}{2}$. Prove that

$$\frac{\sin a+\sin b+\sin c}{\cos a\,(\tan b+\tan c)+\cos b\,(\tan c+\tan a)+\cos c\,(\tan a+\tan b)}\le\frac{1}{2}.$$

Solution *by the proposer*.

Since $0<a\le b\le c<\frac{\pi}{2}$, the vectors $\langle\tan a,\tan b,\tan c\rangle$ and $\langle\cos a,\cos b,\cos c\rangle$ have components that are monotonic. The first sequence of components is non-decreasing, and the second is non-increasing. Since they have reverse order, we can apply Chebyshev's inequality. We get

$$\begin{aligned}&3\,(\cos a\tan a+\cos b\tan b+\cos c\tan c)\\ \le\;&(\cos a+\cos b+\cos c)\,(\tan a+\tan b+\tan c)\end{aligned}$$

or

$$\begin{aligned}3\,(\sin a+\sin b+\sin c)\;\le\;&\sin a+\cos a\tan b+\cos a\tan c\\ &+\cos b\tan a+\sin b+\cos b\tan c\\ &+\cos c\tan a+\cos c\tan b+\sin c.\end{aligned}$$

Therefore,

$$\begin{aligned}2\,(\sin a+\sin b+\sin c)\;\le\;&\cos a\tan b+\cos a\tan c+\cos b\tan a\\ &+\cos b\tan c+\cos c\tan a+\cos c\tan b,\end{aligned}$$

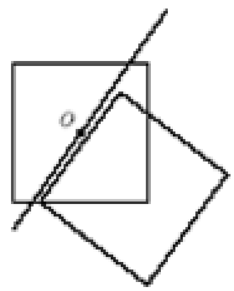
from which the statement follows. Equality holds when $a=b=c$.

Problem 619. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, IL.*

Several identical square napkins are placed on a table. They are placed in such a way that any two of them have a common area which is greater than half of the area of one of them. Is it always possible to pierce all the napkins with a needle going perpendicular to the plane of the table? If yes, prove it. If not, provide a counterexample.

Solution *by the proposers*.

It is always possible. We show that the center of any napkin belongs to every other napkin. Suppose, to the contrary, that one napkin has center O that does not belong to another napkin. Since the other napkin is a square, it is possible to draw a line through O which does not meet the other napkin as illustrated below. Since the line divides the napkin in two equal parts, the second napkin will overlap the first in less than half of the first napkin. This is a contradiction. Hence, in particular, the center of the top napkin belongs to every other napkin and thus a needle through the center of the top napkin will pierce all napkins.



Problem 620. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, IL.*

Nick chooses 81 consecutive integers, rearranges them, and concatenates them to form one long, multi-digit number N. Michael chooses 80 consecutive integers, rearranges them, and concatenates them to obtain the number M. Is it possible that $M = N$? If yes, provide an example. If not, prove it.

Solution *by the proposers*.

Nick chooses the 81 integers from 8 through 88 and Michael chooses the 80 integers 10 through 89. They can both arrange their chosen integers to form the number 891011121314...868788.
Also solved by Tian Cai (student), Pace University, New York, NY.

Problem 621. *Proposed by Lisa Hernandez, Jim Buchholz, Doug Martin (jointly), California Baptist University, Riverside, CA.*

A standard technique for showing that .999... $= 1$ is to let $x =$ .999... and then $10x = 9.999...$ and so $9x = 10x - x = 9$, which gives $x = 1$. Can you derive an alternative proof that .999... $= 1$, perhaps using proof by contradiction?

Solution *by the proposer*.

Suppose that .9999... <1. Then there exists a real number x with .9999... $<x<1$. Since $0<x<1$, we can write $x=0.x_1x_2x_3$... Since .9999... $\neq x$, there must be at least one index i such that $x_i\neq 9$. Then $x_i\in\{0,1,3,...,8\}$. Hence $x<.999$... But now we have a contradiction. Thus it must be that .9999...=1.

Problem 622. *Proposed by the editor.*

Prove that there are infinitely many positive, palindromic integers containing just the digits 2, 7, and 9, which are divisible by 2, 7, and 9.

Solution *by Bill Paulsen, Arkansas State University, Jonesboro, AR*.

Consider the numbers $28\left(10^n-1\right)$ which look like 27999...99972 for $n>1$. Clearly these will be a multiple of 2, 7, and 9 and also will be palindromic numbers containing only the digits 2, 7, and 9. Thus, there are an infinite number of such palindromes.

Solution *by Tian Cai (student), Pace University, New York, NY.*

The number 27972 is a number that satisfies the conditions. Therefore, there exist such numbers. Now suppose there are only finitely many numbers that satisfy the condition. Let X be the greatest. Suppose that X has n digits. Now we create the new number Y such that $Y=X+10^nX$. Obviously, Y also satisfies the conditions, but$Y>X$. This is a contradiction. Therefore, there must be infinitely many positive, palindromic integers containing just the digits 2, 7, 9, which are divisible by 2, 7, and 9. [This proof leads to the sequence 27972, 2797227972, 27972279722797227972,... as a solution.]
Also solved by the proposer.

Problem 623. *Proposed by the editor.*

Let $f\left(n\right)$ be the number of rationals between 0 and 1 (noninclusive) that have denominator less than or equal to n. Prove that $f\left(n\right)>\frac{2}{3}n^{3/2}$ for all integers $n\geq 2$.

Solution *by the proposer*.

Any rational number between 0 and 1 is p/q, with p,q positive integers, $p<q$ and $\gcd\left(p,q\right)=1$. Thus the Euler ϕ function can be used. Since $\phi\left(k\right)$ is the number of integers less than k which are relatively prime to k, $\phi\left(k\right)$ counts the number of rationals between 0 and 1 with denominator

k. This means that $f(n) = \sum_{k=2}^{n} \phi(k)$. Just using the values for $\phi(k)$ with $k = 2, 3, 4, 5$, and 6, we have $\sum_{k=2}^{6} \phi(k) = 11 > 10.832 = \sum_{k=1}^{6} \sqrt{k}$. By the lemma below, we know that $\phi(k) > \sqrt{k}$ for all $k > 6$. Thus, $\sum_{k=2}^{n} \phi(k) > \sum_{k=1}^{n} \sqrt{k}$. Using the type of argument from calculus involving right-hand Riemann sums, we know that $\sum_{k=1}^{n} \sqrt{k} > \int_0^n \sqrt{x}dx = \frac{2}{3}n^{3/2}$. Therefore, $f(n) > \frac{2}{3}n^{3/2}$.

Lemma 1 $\phi(k) > \sqrt{k}$ for all positive integers $k > 6$.

Proof: Let p be a prime. If $a > 1$, then $\phi(p^a) = p^{a-1}(p-1) \geq p^{a-1} \geq p^{a/2} = \sqrt{p^a}$. If p is odd, then $\phi(p) = p - 1 > \sqrt{p}$ and $\phi(2p^a) = p^{a-1}(p-1) \geq 2p^{a-1} \geq \sqrt{2p^a}$. Finally, if $p > 4$, then $p^2 + 1 > 4p$, so $(p-1)^2 = p^2 - 2p + 1 > 2p$, and we have $\phi(2p) = p - 1 > \sqrt{2p}$. Now suppose that $n = 2^a p^b q^c \cdot \cdots$. Consider the following cases.

1. If $a = 0$, then by the multiplicativity of ϕ,

$$\begin{aligned} \phi(n) &= \phi\left(p^b\right)\phi(q^c)\cdots \\ &> \sqrt{p^b}\cdot\sqrt{q^c}\cdots \\ &= \sqrt{n}. \end{aligned}$$

2. If $a = 1$ and $n > 6$, then

$$\begin{aligned} \phi(n) &= \phi\left(2p^b\right)\phi(q^c) \\ &> \sqrt{2p^b}\cdot\sqrt{q^c}\cdots \quad \text{since either } b > 1 \text{ or } p > 3 \text{ since } n > 6 \\ &= \sqrt{n}. \end{aligned}$$

3. If $a > 1$, then

$$\begin{aligned} \phi(n) &= \phi(2^a)\,\phi\left(p^b\right)\phi(q^c)\cdots \\ &= 2^{a-1}\phi\left(p^b\right)\phi(q^c)\cdots \\ &\geq 2^{a/2}\phi\left(p^b\right)\phi(q^c)\cdots \\ &> \sqrt{2^a}\cdot\sqrt{p^b}\cdot\sqrt{q^c}\cdots \\ &= \sqrt{n}. \end{aligned}$$

In all cases, we have the desired inequality.