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The Mathematical Process of Classification

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1. Introduction

The paper will focus on one of the most computationally-intensive types of problems in psychology: classification of individuals among groups. We will generalize to the p-variate case only after demonstrating the effectiveness of the bivariate case. The paper will follow the same basic process that any psychologist might use to increase the success of hiring individuals for any particular type of work. We will start with the predicting equations and then merge into the probability distributions. Once a particular distribution equation is effectively constructed, we can then use this information to establish confidence regions. We will then progress into decision procedures for assigning individuals into groups. This section will explore the probabilities behind classification as well as the probabilities of misclassification.

Often, psychologists are hired to increase the overall efficacy of the hiring procedures for certain corporations. To accomplish this task, the psychologist would first be interested in assigning certain positions a combination of attributes that would hopefully optimize this position's productivity or efficiency. The psychologist will need to consult with the subject-matter experts (SME) to determine which attributes are most important to the company for all of the positions in question. The psychologist needs to prepare an assessment with questions that specifically target those attributes in question. From this, the psychologist can use the scores on the assessment along with corresponding information about that particular person's overall effectiveness or yield. The yield can be hypothetical, but the yield is usually a concrete number like an employee's score on annual evaluations. This number may get more complicated. For instance, if the employer wanted to hire people that would produce a high yield but was willing to sacrifice a high yield for a longer career, then the psychologist would put a 'weight' in the equation so that the yield was higher for individuals that stayed with the company longer (assuming that everything else is equal).

2. Calculating the Yield

Suppose the psychologist has records from a previous similar study in which the data is distributed below. We will call the average of each employee's evaluation the yield or y-component. The x-component will be the scores that the employee achieved on the developed assessment. For graphical purposes, the example will consist of only two attributes or scales. Let the first attribute be verbal ability and the second attribute be mathematical ability.

$$X = \begin{bmatrix} 1 & 100 & 60 \\ 1 & 91 & 69 \\ 1 & 32 & 83 \\ 1 & 50 & 86 \\ 1 & 62 & 58 \\ 1 & 26 & 44 \\ 1 & 93 & 89 \\ 1 & 86 & 82 \\ 1 & 15 & 20 \\ 1 & 92 & 99 \end{bmatrix} \qquad Y = \begin{bmatrix} 67 \\ 73 \\ 70 \\ 74 \\ 59 \\ 32 \\ 100 \\ 97 \\ 12 \\ 100 \end{bmatrix}$$

As mentioned, the psychologist will use the data to predict how well an individual will perform (yield) given his or her score on the developed assessment. To do this, we will fit the simple regression model

$$y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \varepsilon,$$

by using the equation for estimation

$$\widehat{\beta} = \left(X^T X\right)^{-1} X^T y.$$

The $\left(X^TX\right)^{-1}$ matrix is

$$\begin{aligned}
\left(X^TX\right)^{-1} &= \left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 100 & 91 & \cdots & 92 \\ 60 & 69 & \cdots & 99 \end{bmatrix} \begin{bmatrix} 1 & 100 & 60 \\ 1 & 91 & 69 \\ \cdots & \cdots & \cdots \\ 1 & 92 & 99 \end{bmatrix} \right) \\
&= \begin{bmatrix} 10 & 647 & 690 \\ 647 & 51059 & 48712 \\ 690 & 48712 & 52792 \end{bmatrix}^{-1} \\
&= \begin{bmatrix} 1.0372 & -0.0018 & -0.0119 \\ -0.0018 & 0.0002 & -0.0001 \\ -0.0119 & -0.0001 & 0.0003 \end{bmatrix},
\end{aligned}$$

and the X^Ty matrix is

$$X^Ty = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 100 & 91 & \cdots & 92 \\ 60 & 69 & \cdots & 99 \end{bmatrix} \begin{bmatrix} 67 \\ 73 \\ \cdots \\ 100 \end{bmatrix} = \begin{bmatrix} 684 \\ 50795 \\ 53055 \end{bmatrix}.$$

Since the linear regression model will almost never be perfect, we must use the least squares estimate of the vector with the above equation for estimation:

$$\begin{aligned}
\widehat{\beta} &= \left(X^TX\right)^{-1} X^Ty \\
&= \begin{bmatrix} 1.0372 & -0.0018 & -0.0119 \\ -0.0018 & 0.0002 & -0.0001 \\ -0.0119 & -0.0001 & 0.0003 \end{bmatrix} \begin{bmatrix} 684 \\ 50795 \\ 53055 \end{bmatrix} \\
&= \begin{bmatrix} 1.0372 & -0.0018 & -0.0119 \\ -0.0018 & -0.0018 & -0.0001 \\ -0.0119 & -0.0001 & 0.0003 \end{bmatrix} \begin{bmatrix} 684 \\ 50795 \\ 53055 \end{bmatrix} \\
&= \begin{bmatrix} -13.0135 \\ .3231 \\ .8769 \end{bmatrix} = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{bmatrix}.
\end{aligned}$$

Thus, the least squares estimate for this example is

$$\widehat{y} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 = -13.0135 + .3231x_1 + .8769x_2.$$

We might also be interested in the point of maximum yield. If we first write the above equation into matrix notation, we will have

$$y = \beta_0 + x^Tb + x^TBx,$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix};\ b = \begin{bmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_k \end{bmatrix};\ B = \begin{bmatrix} \beta_{11} & \beta_{12}/2 & \cdots & \beta_{1k}/2 \\ \beta_{21}/2 & \beta_{22} & \cdots & \beta_{2k}/2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{k1}/2 & \beta_{k2}/2 & \cdots & \beta_{kk} \end{bmatrix}.$$

The derivative of y with respect to the elements of the vector x when set to 0 will give us the point of maximization or the stationary point.

$$\frac{\partial y}{\partial x} = b + 2Bx = 0 \Longrightarrow x = -\frac{1}{2}B^{-1}b.$$

Now, we can find the predicted response or maximum value (yield) at the stationary point by the equation

$$y = \beta_0 + \frac{1}{2}x^T b.$$

We can use a contour plot to graphically discover the region in which the p-percentile of the scores lies. Having the regression estimate makes the job very nice because after the scores are reported, we can immediately approximate the overall yield that applicant might have. For example, suppose an applicant scores a 76 on verbal ability and an 81 on mathematical reasoning. Given this, we can estimate that individual's potential productivity in the company as being $-13.0135 + .3231(76) + .8769(81) = 82.571$. Because the yield ranges from 100 to 0, we can interpret the applicant's score or predicted yield as being 82.571. Reiterating, if hired, this applicant is predicted to have a yield equal to 82.571 units of yield. Depending of course on the company's hiring status and the distribution of yield among all applicants, they now can consider scheduling an interview for this applicant.

3. Distribution of Yield

Having the predicted yield is only superficially successful. We require a good knowledge of how the rest of the population scores on the same assessment. We acquire this knowledge through the use of statistics. Once we have enough data that we feel sufficiently samples the population, we can determine the probability distribution of the assessment. For both complexity and notation purposes, we will start with the bivariate normal density function given by

$$\phi\left(X_1, X_2\right) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \cdot \exp\left[-\frac{S}{2\left(1-\rho^2\right)}\right],$$

with

$$S = \frac{(X_1 - \mu_1)^2}{\sigma_1^2} + \frac{(X_2 - \mu_2)^2}{\sigma_2^2} - 2\rho \frac{(X_1 - \mu_1)(X_2 - \mu_2)}{\sigma_1 \sigma_2}.$$

Here, σ_i^2 is the variance, μ_i is the mean, and

$$\rho = \frac{\text{cov}(X_1, X_2)}{\sigma_1 \sigma_2} = \frac{E(X_1 - \mu_1) E(X_2 - \mu_2)}{\sigma_1 \sigma_2}$$

is the correlation coefficient. Also note that the equation

$$\frac{(X_1 - \mu_1)^2}{\sigma_1^2} + \frac{(X_2 - \mu_2)^2}{\sigma_2^2} - 2\rho \frac{(X_1 - \mu_1)(X_2 - \mu_2)}{\sigma_1 \sigma_2} = C$$

represents an ellipse with center at point (μ_1, μ_2); this point is known as the centroid of the bivariate population. Later, we will make use of the constant C in finding certain percentiles of distribution given the means for the bivariate population.

To continue to the p-variate case, it is convenient to rewrite the quantities in the equation for the bivariate case in matrix notation. The variance-covariance matrix or the dispersion matrix for the bivariate population can be written as

$$\Sigma_2 = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_2\sigma_1 & \sigma_2^2 \end{bmatrix}.$$

This matrix essentially tells us how the each attribute is related to each other. The dispersion matrix has similar mathematical results as a correlation matrix ([2]). The determinant of matrix is

$$|\Sigma_2| = \sigma_1^2 \sigma_2^2 \left(1 - \rho^2\right),$$

and the inverse of matrix is given by

$$\begin{aligned} \Sigma_2^{-1} &= \frac{1}{\sigma_1^2 \sigma_2^2 (1 - \rho^2)} \begin{bmatrix} \sigma_2^2 & -\rho\sigma_1\sigma_2 \\ -\rho\sigma_2\sigma_1 & \sigma_1^2 \end{bmatrix} \\ &= \frac{1}{(1 - \rho^2)} \begin{bmatrix} \frac{1}{\sigma_1^2} & \frac{-\rho}{\sigma_1\sigma_2} \\ \frac{-\rho}{\sigma_2\sigma_1} & \frac{1}{\sigma_2^2} \end{bmatrix}. \end{aligned}$$

We can rewrite $\phi(X_1, X_2)$ more compactly as

$$\phi(X_1, X_2) = (2\pi)^{-1} |\Sigma_2|^{-1/2} \exp\left(-\frac{\chi^2}{2}\right),$$

where $\chi^2 = x^T \Sigma_2^{-1} x$, and $x^T = [X_1 - \mu_1, X_2 - \mu_2]$.

We can now progress into the p-variate case with the new terms all in their p-variate sense:

$$\phi\left(X_1, X_2, ..., X_p\right) = (2\pi)^{-p/2} \left|\Sigma_p\right|^{-1/2} \exp\left(-\frac{\chi^2}{2}\right) \sim N\left(\mu, \Sigma\right),$$

with the hyperellipsoid of the distribution being given by

$$\chi^2 = x^T \Sigma_p^{-1} x = C.$$

4. Using the Distribution

Once we have enough data to properly establish a probability distribution, we can then supply confidence regions to aid in the classification of applicants. For example, if we wanted our applicants to score in the 10th percentile of scores for a particular trait, we could look up the value that corresponded to .90. In this case, $P\left(\chi^2 \leq 4.605\right) = .90$. From this we can denote by $R_2\left(4.605\right)$ the region bounded by and including the ellipse

$$\frac{1}{1-p^2}\left[\frac{\left(X_1-\mu_1\right)^2}{\sigma_1^2} + \frac{\left(X_2-\mu_2\right)^2}{\sigma_2^2} - 2\rho\frac{\left(X_1-\mu_1\right)\left(X_2-\mu_2\right)}{\sigma_1\sigma_2}\right] = 4.605,$$

where the subscript of R indicates the bivariate case. The geometrical interpretation of this is if we were to take many observations and assign them values $\left(X_1, X_2\right)$, then 90% of the scores would lie on or in the region $R_2\left(4.605\right)$.

We can use this technique to determine other useful information such as the likelihood that any particular score has in the population. The following data is arbitrary. Consider a population in which

$$\mu = \begin{bmatrix} 70 \\ 65 \end{bmatrix} \text{ and } \Sigma_2 = \begin{bmatrix} 225 & 182 \\ 182 & 230 \end{bmatrix}.$$

Suppose we are interested in the 80th percentile; that is, the equation of the ellipse inside of which or on which 20% of the population exists. Here, the 20% mark is $P\left(\chi^2 \leq .4463\right) = .20$. From the equation we know that the population has the distribution

$$\begin{aligned} & x^T\Sigma^{-1}x \\ = \; & \frac{1}{1-.8^2}\left[\frac{\left(X_1-70\right)^2}{225} + \frac{\left(X_2-65\right)^2}{230} - 1.6\frac{\left(X_1-70\right)\left(X_2-65\right)}{15\cdot 15.17}\right] \\ = \; & .4463. \end{aligned}$$

This equation represents an ellipse in which the set of X_1 and X_2 that satisfy the above equation create the outline of the ellipse. Note that this procedure can be reversed to find the likelihood of a particular score. For example, if an applicant scores a 90 and a 95 respectively, then using the χ^2 distribution table with two degrees of freedom, we find that this point lies on the ellipse representing the 87.05 percentile. This means that 12.95% of all applicants lie within – geometrically closer to the centroid or exactly the same distance – the region generated by the ellipse passing through the point (90, 95) and 87.05% of all applicants fall on the outside of this ellipse.

5. Applying the Calculations

It is important to note that the percentile of the applicant does not give us an indication of his or her abilities. We must make a few more computations to be able to actually use this statistic. Consider Applicant A, who scores a low score on both attributes. For arguments sake, Applicant A's score is in the 10th percentile (again, that is 90% of the population score the same or geometrically closer to the population mean). Furthermore, Applicant B, who scores very well on both attributes also scores in the 10th percentile. Here it is clear that the percentile simply shows the likelihood of the particular score, not the aptitude. To find this, we should consider the regression model we produced in the beginning of the paper.

Now that we have developed these two methods, we can determine what percent of the population has a yield that falls within a range of scores. This is a useful technique because we can set realistic expectations for the people taking our assessment. For example, if we wanted to know how unlikely it would be for a person to score a yield of 80 or above on our assessment, we simply find the volume of the multivariate normal distribution with the region on the x_1x_2-plane being the cross-section of the regression model that corresponds to yield of 80. We use the formula

$$\iint\limits_A \phi\left(X_1, X_2\right) dx_1 dx_2,$$

where

$$A = \left\{\left(x_1, x_2\right) | \beta_0 + \beta_1 x_1 + \beta_2 x_2 = 80\right\}.$$

For the p-variate case,

$$\int \underset{A}{\cdots} \int \phi\left(X_1, X_2, ..., X_p\right) dX_1 \cdots dx_p,$$

where

$$A = \left\{ (x_1, x_2, ..., x_p) \left| \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_p x_p = 80 \right. \right\}.$$

6. Classification

Now we will focus more on the idea of classifying applicants based on his or her score on the assessment. In classification, we are interested in finding the minimum distance from a given score to a group's region, π_i. This makes intuitive sense in that a given score vector is closest to a particular mean. Restating, we will be considering any particular score vector, x, and determining which group the applicant should be assigned to based on the statistical distance the score is from the centroid of the group distributions.

This region can be altered by *a priori* probabilities, and by the cost of misclassification. *A priori* probabilities are probabilities that are logically determined from existing information. We would likely use *a priori* probabilities to account for our need for particular positions. For example, the accounting department might only staff eight people, but the sales department might staff 30 people. In this simple example, the priori probability of the accounting department is 8/38 and the *a priori* probability of the sales department is 30/38. This type of alteration can be used based on the number of people we are hiring for each position. However, it is not necessary to know relative staffing information in every instance.

The cost of misclassification is simply the cost of classifying an individual incorrectly. Consider two positions, an accounting position and a cashiering position. The cost of misclassifying a potential accountant as a cashier is not as high as the cost of misclassifying a cashier as an accountant. The misclassified accountant might be unskilled in financial mathematics and would thus hinder the company more than an individual who is simply not meeting his or her fullest potential (a misclassified cashier that could have been an accountant). Such costs are usually determined by company guidelines and then used computationally by the psychologist.

To find each classification region, we will first need to compare the ratio of the densities of the two multivariate normal populations:

$$\frac{\phi(X_1, X_2)}{\phi(Y_1, Y_2)} \to \frac{\phi_1(x)}{\phi_2(x)} = \frac{\exp\left(-\frac{\chi_1^2}{2}\right)}{\exp\left(-\frac{\chi_2^2}{2}\right)} = \exp\left[-\frac{1}{2}\left(\chi_1^2 - \chi_2^2\right)\right].$$

Note the change in notation. For this model, we will assume that the two populations have equal variance-covariance matrices, $N(\mu_1, \Sigma)$ and $N(\mu_2, \Sigma)$. The region of classification is the set of x's that make the above equation greater than or equal to k ([1]). We choose the logarithmic function because it is monotonically increasing; thus, the resulting inequality becomes

$$-\frac{1}{2}\left(\chi_1^2 - \chi_2^2\right) \geq \log k.$$

As a consequence, the best regions of classification are given by

$$\begin{aligned} R_1 &: -\frac{1}{2}\left(\chi_1^2 - \chi_2^2\right) \geq \log k \\ R_2 &: -\frac{1}{2}\left(\chi_1^2 - \chi_2^2\right) < \log k. \end{aligned}$$

If *a priori* probabilities q_1 and q_2 are known, then k is given by

$$k = \frac{q_2 C(1|2)}{q_1 C(2|1)},$$

where $C(i|j)$ is the cost of misclassification into group i given that it was taken from or belongs to j. Otherwise, $\ln k = c$, for some c that is suitably chosen.

To determine the probability of misclassification with two groups, we will use the Mahalanobis squared distance between $N(\mu_1, \Sigma)$ and $N(\mu_2, \Sigma)$:

$$\Delta^2 = (\mu_1 - \mu_2)^T \Sigma^{-1} (\mu_1 - \mu_2).$$

From this, we find that the distribution of the ratio of the densities is $N\left(\frac{1}{2}\Delta^2, \Delta^2\right)$. If the costs of misclassification are equal, then the probability of misclassification becomes

$$\int_{\Delta/2}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy.$$

Now, we will progress into the case of classification into several multivariate normal populations. For now, we will assume that the means and the variance-covariance matrices are different but the cost of misclassification for each group is equal. We will use the Mahalanobis squared distance for the general case.

Now, to classify simply means to find the minimum distance between a raw score and a mean score. We obtain this by comparing the distances $\Delta_i^2(x)$ for each π_i. We will use the discriminant function to classify x to R_i when the linear discriminant score is largest:

$$d_i(x) = (\mu_i)^T \Sigma^{-1} x - \frac{1}{2}(\mu_i)^T \Sigma^{-1} (\mu_i) + \ln(p_i).$$

We can use the above equation to compare two scores at the same time. The term $\ln\left(\frac{p_2}{p_1}\right)$ is meant to place the plane of separation closer to μ_1

than μ_2 if p_2 is greater than p_1. We assign x to R_k if

$$(\mu_k - \mu_i)^T \Sigma^{-1} x - \frac{1}{2} (\mu_k - \mu_i)^T \Sigma^{-1} (\mu_k + \mu_i) \geq \ln \left(\frac{p_i}{p_k} \right) \quad i = 1, 2, ..., n.$$

For example, if we want to partition a space into three regions ($n = 3$), we have only to define the regions so that consists of all x satisfying

$$R_1 : (\mu_k - \mu_i)^T \Sigma^{-1} x - \frac{1}{2} (\mu_k - \mu_i)^T \Sigma^{-1} (\mu_k + \mu_i) \geq \ln \left(\frac{p_i}{p_k} \right), i = 2, 3.$$

Now we will create an example of classifying an individual into one of three groups. Let the groups consist of accountants (π_1), cashiers (π_2), and managers (π_3). Let an assessment be developed so that it measures verbal ability (x_1), mathematical ability (x_2), and integrity (x_3). Suppose the following table has modeled a company for many years, and acquired a large value of N. The mean scores are shown below.

	Accountants (π_1)	Cashiers (π_2)	Managers (π_3)
Verbal Ability (x_1)	63	85	81
Math Ability (x_2)	96	61	70
Integrity (x_3)	85	43	87

$$\Sigma = \begin{bmatrix} 25 & 18 & 19 \\ 16 & 27 & 17 \\ 13 & 15 & 23 \end{bmatrix} \qquad \begin{array}{l} p_1 = .25 \\ p_2 = .65 \\ p_3 = .1 \end{array}$$

For actual results, we need a raw score. Suppose an applicant's score vector is shown below.

$$\begin{bmatrix} 65 & 91 & 79 \end{bmatrix}^T.$$

With this, we need only to make the calculations to find the minimum distance:

$$
\begin{aligned}
d_{13}(x) &= (\mu_1 - \mu_3)^T \Sigma^{-1} x - \frac{1}{2}(\mu_1 - \mu_3)^T \Sigma^{-1} (\mu_1 + \mu_3) \geq \ln\left(\frac{p_3}{p_1}\right) \\
36.8 &\geq -.9163 \\
d_{12}(x) &= (\mu_1 - \mu_2)^T \Sigma^{-1} x - \frac{1}{2}(\mu_1 - \mu_2)^T \Sigma^{-1} (\mu_1 + \mu_2) \geq \ln\left(\frac{p_2}{p_1}\right) \\
117 &\geq .9555 \\
d_{23}(x) &= (\mu_2 - \mu_3)^T \Sigma^{-1} x - \frac{1}{2}(\mu_2 - \mu_3)^T \Sigma^{-1} (\mu_2 + \mu_3) \geq \ln\left(\frac{p_3}{p_2}\right) \\
-68.95 &\leq -1.872 \\
d_{21}(x) &= (\mu_2 - \mu_1)^T \Sigma^{-1} x - \frac{1}{2}(\mu_2 - \mu_1)^T \Sigma^{-1} (\mu_2 + \mu_1) \geq \ln\left(\frac{p_1}{p_2}\right) \\
-117 &\leq -.9555 \\
d_{31}(x) &= (\mu_3 - \mu_1)^T \Sigma^{-1} x - \frac{1}{2}(\mu_3 - \mu_1)^T \Sigma^{-1} (\mu_3 + \mu_1) \geq \ln\left(\frac{p_1}{p_3}\right) \\
-36.8 &\leq .9163 \\
d_{32}(x) &= (\mu_3 - \mu_2)^T \Sigma^{-1} x - \frac{1}{2}(\mu_3 - \mu_2)^T \Sigma^{-1} (\mu_3 + \mu_2) \geq \ln\left(\frac{p_2}{p_3}\right) \\
68.95 &\geq 1.872
\end{aligned}
$$

From the calculations, we see a few things: $d_{ij} = -d_{ji}$, and x only satisfied one pair of equations, namely d_{13} and d_{12}. This can be interpreted to mean that the distances from the x provided to region one are smallest in this situation. Therefore, we make the classification that this applicant would be best suited as an accountant. We can then take this information and calculate what percentage of the population scores closer to the mean for accountants and make an inference as to whether or not we should consider this individual for employment. Just because this applicant is best-suited to be an accountant, does not necessarily mean that this applicant is the best potential applicant for the job.

In this paper, we have seen the mathematics involved in the delicate procedure of classification. It is important to note that there have been many different procedures designed to further analyze the ideas discussed here. To construct the paper, we have examined some of these procedures but decided to only use those selected. Additional works will analyze the effectiveness of the remaining procedures as well as showing the calculations for the distribution of the yield statistic using the procedures (given in [1]) to their full extent.

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Musical Mathematics

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1. Introduction

This paper will address some of the similarities of serialistic music and mathematics. The basics of music must first be established in order to understand serialistic music. Then we can look at some of the mathematical correlations.

In music, a single tone is defined as a pitch, which is represented by a letter. There are many pitches that one can hear, but after removing duplicates, there are twelve distinct pitches. The different pitches are represented as follows: A, B♭, B, C, D♭, D, E♭, E, F, G♭, G, A♭. The ♭ after the letter means that the pitch is flat; a ♯ would mean that the pitch is sharp. There are multiple ways to spell the same pitch; for example: A♯ is the same as B♭. There is a ♭ or # between every letter except between B and C and between E and F. These are all of the different pitches because this sequence is a continuous loop. After G is A♭, then A, and so on. For the purposes of this paper, we are only concerned with one set of these pitches.

The distance from one letter to the next, A to B♭, is called a half step or a minor second, denoted m2. The distance from A to B is called a whole step or major second, denoted M2. Here is the series of intervals from smallest to largest: m2, M2, m3, M3, P4, TT, P5, m6, M6, m7, M7. The P4 stands for perfect fourth and the TT represents a tritone, otherwise known as an augmented fourth or minor fifth. These are all of the intervals we will need to use in serialistic music. In order to take the inversion of an interval, we determine the distance and direction that it occurs and go the same distance in the opposite direction. For example, the inversion of a m2 from A to B♭ is a m2 but from A to A♭. Now that we have a basic understanding of music, we can describe serialistic music.

In order to understand serialistic music, it is necessary to explain why this type of music was written in the first place. Before and during World War 2, almost all music was based on the work of classical composers, following their style of harmonization and chord progressions. In short, each piece had a standard set of rules to follow that made it sound acceptable. Tonal music was a very popular method used during this time period. Tonal music is music that is centered around one particular pitch, which is termed the tonic. German composers used tonal music for many of their military actions, so following the war, there was a desire by many countries to hear a new type of music. Serialistic music was one solution to that desire. Serialistic music is atonal music, which means that the music is not centered around any one individual pitch. Some composers took this even further and intentionally wrote music that used all twelve pitches equally. One of these composers was Arnold Schoenberg, who decided to take serialism further than others did by creating a twelve-tone matrix.

2. Twelve-tone matrix

A twelve-tone matrix consists of twelve rows and twelve columns. Each row contains all twelve different pitches with no repeats. The same is true for the columns. Each row is essentially the same but starts on a different pitch; the intervals between the first and second note in each row are the same. In order to construct a twelve-tone matrix, we decide the order in which we want the first row to appear. It does not matter what the order is, but it is usually a mixed or random order, so there is no musicality affecting the decision. For the purposes of this paper, let's assume that the order for the first row is as follows: B♭, F, C, B, A, G♭, D♭, E♭, G, A♭, D, E. Now that we have established the first row of the twelve-tone matrix, we must assign a numerical value to each pitch. For this, we will use modular arithmetic with 12 as our modulus. Each pitch will have a number from 0 to 11 assigned to it. When assigning numbers for a row, the first pitch will be labeled as 0, and then the next pitch, up by a m2, will be 1, and so on, until every number is assigned. For our first row, the assignment of numbers will be as follows: B♭ = 0, F = 7, C = 2, B = 1, A = 11, G♭ = 8, D♭ = 3, E♭ = 5, G = 9, A♭ = 10, D = 4, E = 6.

The first column

The next step in establishing the twelve-tone matrix is to set up the first column. The first column is the set of inverses. There are two ways to establish what each inverse is for the first column. One method is to look at the intervals of the first row: From B♭ to F is a P5 up or a P4 down, so the inverse will be either a P5 down or a P4 up, which is E♭. That will be the second pitch in the first column.

For the next pitch, we look at the next interval in the first row. That is F to C, which is also a P5 up, so by going a P5 down from the E♭ in the first column we get A♭. This procedure makes the interval from the first to second pitch in the first row the inverse of the interval from first to second pitch in the first column. Using this method to complete the first column, the interval between any two pitches in the first row will be the inverse of the interval between the same locations of pitches in the first column. The first three pitches of the first column are now completed, as the first pitch of the first row and the first pitch of the first column are the same.

Moving on to the fourth pitch, the interval from C to B is up a M7, so by going down a M7 from A♭, we get A as the fourth pitch in the first column. For the fifth pitch, the interval from B to A is up a m7; so down a m7 from A is B.

For the sixth pitch, A to G♭ is up a M6; down a M6 from B is D. For the seventh pitch, G♭ to D♭ is up a P5; down a P5 from D is G. For the eighth pitch, D♭ to E♭ is up a M2; down a M2 from G is F. For the ninth pitch, E♭ to G is up a M3; down a M3 from F is D♭. For the tenth pitch, G to A♭ is up a m2; down a m2 from D♭ is C. For the eleventh pitch, A♭ to D is up a TT; down a TT from C is G♭. Finally, for the twelfth pitch, D to E is up a M2; down a M2 from G♭ is E. As we can see, this method is rather time consuming and a quicker method would be nice to use.

The second method is to look at the numbers assigned to the pitches. In order to construct the first column using this method each number must be paired with its inverse. Since we are using modular arithmetic in $\bmod 12$, the inverse of any number is the corresponding number that can add to the number to get 0. In $\bmod 12$, 0 and 12 are equivalent, so we can find the inverse by taking 12 minus the number. Now in order to find the first column, we take 12 minus the value of the pitch in the same spot in the first row. For example, to find the first pitch of the first column, we take 12 minus the value of the first pitch of the first row, which is 0, and $12 - 0 = 12$ or 0 in $\bmod 12$, so the first pitch of the first column is the pitch represented by the value 0, which is B♭.

Here is how it works out for the first column using this method. For the second pitch, 12 – 7 = 5, which is E♭; for the third pitch, 12 – 2 = 10, which is A♭; for the fourth pitch, 12 – 1 = 11, which is A; for the fifth pitch, 12 – 11 = 1, which is B; for the sixth pitch, 12 – 8 = 4, which is D; for the seventh pitch, 12 – 3 = 9, which is G; for the eighth pitch, 12 – 5 = 7, which is F; for the ninth pitch, 12 – 9 = 3, which is D♭; for the tenth pitch, 12 – 10 = 2, which is C; for the eleventh pitch, 12 – 4 = 8, which is G♭; and for the twelfth pitch, 12 – 6 = 6, which is E. This method also shows us what the inverse for each pitch is. No matter which method we use to find the first column, we get the same result, which is B♭, E♭, A♭, A, B, D, G, F, D♭, C, G♭, E.

Completing the matrix

The first row and first column are now completed; with this information, we can complete the twelve-tone matrix. Again, there are two different ways to do this. They are similar, except one method uses the pitches, and the other looks at the value of the pitches. Here is the method using the pitches. Since our first row started with B♭, we need to find the row that is going to start with the pitch up a m2 from B♭, which is B. Once we have found this row, every pitch from our first row will be copied from its position to that row; however, they are all moved up a m2. The row starting with B would be constructed as follows: F up a m2 is G♭, C up a m2 is D♭, B up a m2 is C, A up a m2 is B♭, G♭ up a m2 is G, D♭ up a m2 is D, E♭ up a m2 is E, G up a m2 is A♭, A♭ up a m2 is A, D up a m2 is E♭, and E up a m2 is F. This method is quick if we have a good understanding of the order of the pitches and intervals. We continue this process by moving to the row that starts with C and taking everything from the row starting with B and moving it up a m2. We do this until every row is completed.

The other method is to look at the values of the pitches. When doing this, we start at any row we want. Let's start with next row, which begins with E♭. Since E♭ is also 5, we can add 5, in $\bmod 12$, to all of the values to our first row to fill out that row. So, the row would look like this: 0 + 5 = 5, which is E♭; 7 + 5 = 12, which is B♭; 2 + 5 = 7, which is F; 1 + 5 = 6, which is E; 11 + 5 = 16 = 4, which is D; 8 + 5 = 13 = 1, which is B; 3 + 5 = 8, which is G♭; 5 + 5 = 10, which is A♭; 9 + 5 = 14 = 2, which is C; 10 + 5 = 15 = 3, which is D♭; 4 + 5 = 9, which is G; and 6 + 5 = 11, which is A. We can use either of these methods to complete the rest of the twelve-tone matrix. The completed twelve-tone matrix is shown. It is the same matrix used in Schoenberg's Op. 33a.

	0	7	2	1	11	8	3	5	9	10	4	6
0	B♭	F	C	B	A	G♭	D♭	E♭	G	A♭	D	E
5	E♭	B♭	F	E	D	B	G♭	A♭	C	D♭	G	A
10	A♭	E♭	B♭	A	G	E	B	D♭	F	G♭	C	D
11	A	E	B	B♭	A♭	F	C	D	G♭	G	D♭	E♭
1	B	G♭	D♭	C	B♭	G	D	E	A♭	A	E♭	F
4	D	A	E	E♭	D♭	B♭	F	G	B	C	G♭	A♭
9	G	D	A	A♭	G♭	E♭	B♭	C	E	F	B	D♭
7	F	C	G	G♭	E	D♭	A♭	B♭	D	E♭	A	B
3	D♭	A♭	E♭	D	C	A	E	G♭	B♭	B	F	G
2	C	G	D	D♭	B	A♭	E♭	F	A	B♭	E	G♭
8	G♭	D♭	A♭	G	F	D	A	B	E♭	E	B♭	C
6	E	B	G♭	F	E♭	C	G	A	D♭	D	A♭	B♭

Interpreting the matrix

There are four different ways to read this twelve-tone matrix. The matrix read from left to right across the rows is called the prime form, denoted P-#, where # is the value of the first pitch in that row. For example, if we read the row starting with B♭, it would be denoted P-0. Another way to read the matrix is to read the rows from right to left. Reading the matrix in this way is termed the retrograde, denoted R-#, where # is the value of the far left pitch of the row. If we were to read R-0, it would have B♭, since B♭ is 0, on the far left, even though we would read R-0 from right to left. The other two ways to read the matrix are to read the columns. One way to read the columns is to read them from top to bottom, which is called the inversion and denoted by I-*, where * is the value of the first pitch in that column. Thus, I-0 would be the column with B♭ at the top. The final way to read the matrix is from bottom to top, which is called the retrograde inversion, and denoted RI-*, where * is the value of the top pitch in the column. RI-0 would be the column with the top pitch being B♭, even though we read RI-0 from bottom to top.

3. Use of the Twleve-tone matrix

When composers, like Schoenberg, used this matrix for composing a piece, they typically used a rhythm that was separated like the pitches. Since there is no tonal center, there is no need to use a rhythm to stress any particular pitch. Therefore, the rhythm was often not very focused on the beat, thereby giving the piece a stronger feeling of disconnect with the classical style of composing. Schoenberg used this method when writing a piano sonata. He used a couple of different methods for his composing. For one method, he gave each part a separate row form, so that four row forms were going on at the same time. He also only used two row forms at one time by having one hand play one row form and the other hand play the other row form. Sometimes the row forms would be complete in just one measure, sometimes as quick as a half of a measure and sometimes a single row form would take several measures before it was completed.

Schoenberg was one of many composers that used this method; Antonio Weber is another such composer. Unlike Schoenberg, though, Weber wrote for an orchestra but was more restrictive in his method. Not only did he use the row forms, but in one of his pieces, he also only used twelve different pitches at a time; he did not use any other octaves. Another composer of this method is Pierre Boulez; he took the use of the matrix further than any other composer did. Boulez used the matrix to determine what pitch to play, but he also used it to determine the rhythm, dynamics, and the articulation. When he did this, he set up twelve different rhythms, dynamics, and articulations and assigned them a value. He also separated his twelve-tone matrix into two matrices, one for the prime rows, and one for inversion columns. He then used different methods for reading these matrices to determine the pitch, durations, dynamics, and articulation.

4. The Twelve-tone matrix as a group table

Using the twelve-tone matrix established earlier in this paper, we can see that this matrix contains some mathematical properties. Through mathematical reasoning, we can verify that a twelve-tone matrix is a group table, with the set of twelve pitches forming a group. A group is, by definition, a non-empty set, together with a binary operation, that: is closed, is associative, has an identity, and for which each element has an inverse. We will go through the steps of proving that the set is a group.

First, consider that in order to be a group, the set must be nonempty and equipped with a binary operation. The set is nonempty, as we have defined our set to consist of the twelve distinct pitches, which can be represented in either letter or numerical notation. The operation is addition. This can be viewed in two different ways, but it is still addition. We can either look at the numerical values and show the addition or the letter notation and add the intervals.

Closure

The next step is to prove that this set is closed. We can observe this in either numerical or letter notation. In number notation, we are using modular arithmetic $\bmod 12$. Taking any two values from our set and combining them, we see that our result is still between 0 and 11. Alternatively, in letter notation, we pick any two of the twelve pitches and combine their intervals, always obtaining one of the twelve pitches.

Associativity

Now we must show that this set has the property of being associative. This follows from the associativity of addition. We can illustrate the associativity in both the number and the letter notation. In numerical notation, associativity means that for any three numbers x, y, and z in our set,

$$(x + y) + z = x + (y + z).$$

In letter notation, we look at the locations of the intersection in the matrix. For example C + D would be where the row starting with C and the column starting with D intersect, which is E, so C + D = E. We can take B♭ + (C + D) = B♭ + E, since B♭ is our starting pitch, then B♭ + E = E, similarly (B♭ + C) + D = C + D, which is E. So,

$$\mathrm{B}\flat + (\mathrm{C} + \mathrm{D}) = (\mathrm{B}\flat + \mathrm{C}) + \mathrm{D}.$$

Identity

Finding the identity element is the next step in the process of proving that the set is a group. An identity element is an element that when added to any element will not change its value. In numerical notation, our set consists of the numbers: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11. The identity element is 0 because if we take $0 + x$ we get x and if we take $x + 0$ we also get x. Therefore, 0 is the identity element. In the matrix, this corresponds to B♭.

Inverses

The final process is to show that each element has an inverse. In other words, given an arbitrary element in our set, we must show that there is some element in the set that when combined with the original element gives the identity element. Showing this with numbers in our set means that, for a given element in our set, we need to find an element that we can add to that element to get 0, since that is our identity element. For example, if we take C which has a numerical value of 2 we would need to combine it with A♭, which has a numerical value of 10. When we combine these two elements we get B♭, which has a numerical value of 0. We can also check to see if this is correct by starting with the row beginning with either A♭ or C and find the column that starts with C or A and see if there intersection is B♭. This is true as we find that the intersection of P-10 and I-2 is B♭, similarly the intersection of P-2 and I-10 is also B♭. If we were to list each element with its inverse, it would look like this:

Element	Inverse
B♭	B♭
B	A
C	A♭
D♭	G
D	G♭
E♭	F
E	E
F	E♭
G♭	D
G	D♭
A♭	C
A	B

Since each element has an inverse this shows us that this set is a group by definition. This group however is not just a group; it is an abelian group. An abelian group is a group that meets all of the conditions to be a group and is also commutative.

Commutativity

For a group to be commutative it means that it does not matter in which order the operation occurs. There are two ways to show this. Using the numerical interpretation, we need to verify that for any two elements x and y in our set, $x + y = y + x$; this follows from commutativity of addition. Using the matrix, we pick any two pitches, x and y, and show that the intersection of the row starting with x and the column starting with y is the same pitch as the intersection of the row starting with y and the column starting with x. We can readily verified that this holds in all cases. For example consider D and G. The intersection of the row starting with D and the column starting with G is B, as is the intersection of the row starting with G and the column starting with D.

5. The number of twelve-tone matrices

There are many different possible twelve-tone matrices, but exactly how many? Since the first row determines the entire matrix, and the same pitch does not appear twice in the row, there are $12! = 479,001,600$ for all possible matrices. However, each matrix has 47 other matrices that will look the same based on mirror images, and rotations.

Since the matrix can be read in four different ways, the number of possible matrices reduces. If we make P-2 our new P-0 and generate a new matrix, it will be the same as the matrix generated by P-0 but in a different order. This means that the order of the intervals between pitches matters for distinctness. With this information, we can show how many duplicate matrices will occur for one particular matrix.

There are two cases for determining the number of duplicates. Since the order of the intervals matters, if the intervals appear in a palindrome pattern then there will be the same number of duplicate matrices but some of the rows and columns will be exactly the same, so they will not need to be counted twice. First, we need to determine how many palindromes are possible. In order to be a palindrome the first interval can be anything, but the last interval must be the same as the first. So, the number of palindromes is $11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1,663,200$, since we can not repeat any of the same pitch. Taking the total number of matrices minus the number of palindrome matrices gives $479,001,600 - 1,663,200 = 477,338,400$ matrices that are not palindromes.

When working with the non-palindrome matrices for each matrix, there are 47 other matrices that look the same. By taking the number of matrices divided by 48, we get the number of distinct non-palindrome matrices, $477,338,400/48 = 9,944,550$.

When working with the palindrome matrices, we will have 23 duplicate matrices for each matrix because the rows are repeated as columns. This gives us $1,663,200/24 = 69,300$ palindrome matrices.

By adding these two sets, we will get the number of distinct matrices, $9,944,550 + 69,300 = 10,013,850$. This is far less than the total number of matrices. With this information we can tell that there are $479,001,600 - 10,013,850 = 468,987,750$ matrices that are just duplicates of other matrices.

6. Conclusion

In conclusion, we can see that there is a lot of mathematics present in the music world. There are many other instances of math occurring with music as well, like the physics of sound, the math of counting a rhythm or subdividing, or even devising a meter. This was just a look at how mathematics fits in with the serialistic composition techniques of the twentieth century.

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An Alternative Method of Linear Regression

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1. Introduction

Modeled on the Method of Least Squares, this is the explanation of a slightly different method of linear regression. The Method of Least Squares finds the regression line that minimizes the sum of squared residuals. Where the residual error is calculated by finding the vertical distance from the point to the regression line. Our proposal calculates residual error by taking the perpendicular distance from the point to the regression line. The resulting line is similar in some ways to the usual line, but there are some striking differences.

2. Standard Least Squares

We are looking to find the regression line of the form

$$y = ax + b.$$

In addition, we need to find the a and b that will minimize the vertical distance

$$e_i = Y_i - (aX_i + b),$$

for each point of our data set. The quantity we wish to minimize is the sum of squares of the e_i's:

$$SS = \sum_{i=1}^{n} e_i^2 = \sum_{i=1}^{n} \left(Y_i - (aX_i + b)\right)^2.$$

We do this by taking the partial derivatives of SS with respect to a and b, setting equal to 0, and solving. Doing this for the partial derivative with respect to b we find:

$$\frac{\partial SS}{\partial b} = \sum_{i=1}^{n} 2\left(Y_i - (aX_i + b)\right)(-1) = 0.$$

Simplifying gives

$$-2\sum_{i=1}^{n} (Y_i - aX_i - b) = 0,$$

which reduces to

$$\sum_{i=1}^{n} Y_i = a\sum_{i=1}^{n} X_i + nb$$

and to

$$b = \frac{\sum\limits_{i=1}^{n} Y_i}{n} - \frac{a\sum\limits_{i=1}^{n} X_i}{n}.$$

Let $\overline{Y}$ denote $\frac{\sum_{i=1}^{n} Y_i}{n}$ and $\overline{X}$ denote $\frac{\sum_{i=1}^{n} X_i}{n}$ so the equation can further be simplified to $b = \overline{Y} - a\overline{X}$.

When solving in respect to b, notice that the regression line passes through the centroid of the points (X_i, Y_i).

Repeating the process for a, we obtain:

$$\frac{\partial SS}{\partial a} = \sum_{i=1}^{n} 2\left(Y_i - (aX_i + b)\right)(-X_i) = 0,$$

which simplifies to

$$-2\sum_{i=1}^{n} X_iY_i - aX_i^2 - bX_i = 0,$$

and finally to

$$\sum_{i=1}^{n} X_iY_i = a\sum_{i=1}^{n} X_i^2 + b\sum_{i=1}^{n} X_i.$$

Substituting in for b gives

$$\sum_{i=1}^{n} X_iY_i = a\sum_{i=1}^{n} X_i^2 + \left(\sum_{i=1}^{n} Y_i - a\sum_{i=1}^{n} X_i\right)\sum_{i=1}^{n} X_i,$$

which reduces to

$$a\left(\sum_{i=1}^{n} X_i^2 - \frac{\left(\sum\limits_{i=1}^{n} X_i\right)^2}{n}\right) = \sum_{i=1}^{n} X_i Y_i - \frac{\left(\sum\limits_{i=1}^{n} X_i\right)\left(\sum\limits_{i=1}^{n} Y_i\right)}{n}.$$

We define some new quantities SS_X, SS_Y, and SS_{XY} by

$$SS_X = \sum_{i=1}^{n} X_i^2 - \frac{\left(\sum\limits_{i=1}^{n} X_i\right)^2}{n},$$

$$SS_Y = \sum_{i=1}^{n} Y_i^2 - \frac{\left(\sum\limits_{i=1}^{n} Y_i\right)^2}{n},$$

and

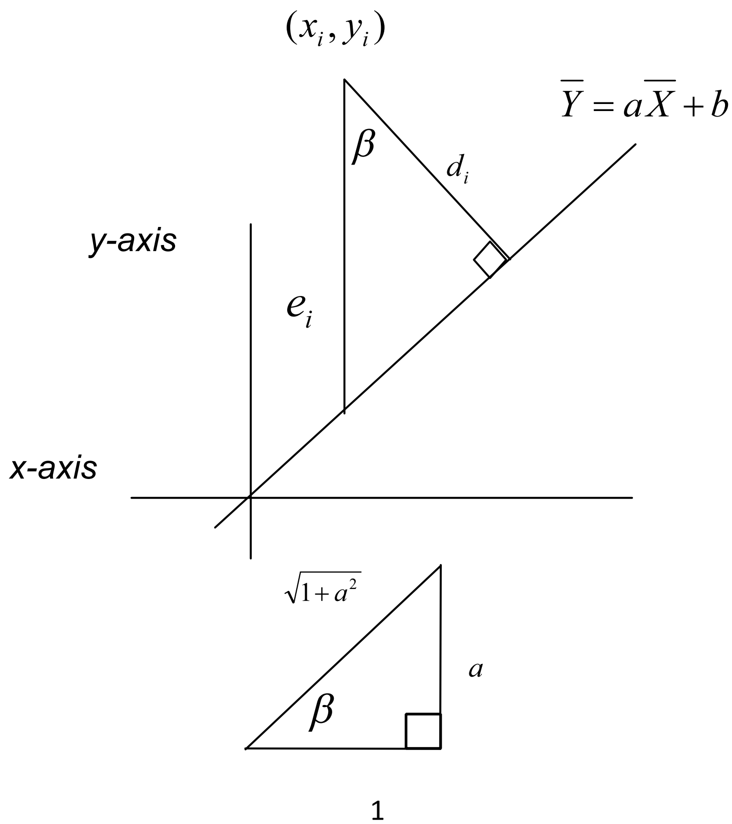
$$SS_{XY} = \sum_{i=1}^{n} X_i Y_i - \frac{\left(\sum\limits_{i=1}^{n} X_i\right)\left(\sum\limits_{i=1}^{n} Y_i\right)}{n}.$$

Then the final equation for the slope of the method of least squares is

$$a = \frac{SS_{XY}}{SS_X}.$$

3. Alternative Regression

The slope for the alternative method is derived using the perpendicular distance instead of the vertical distance parallel to the y-axis. The following picture depicts the difference between the two regressions.



Note that e_i is the residual error for Least Squares and d_i is the residual error that this alternate method minimizes. Also, d_i is the perpendicular distance to the regression, and by using similar triangles, we find that $\frac{d_i}{e_i} = \frac{1}{\sqrt{1+a^2}}$. Solving this equation for d_i and squaring both sides gives $d_i^2 = \frac{e_i^2}{1+a^2}$. The sum of squares of the d_i's is

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} \frac{e_i^2}{1+a^2}.$$

The partial derivative with respect to b will be unaffected by the multiplier $\frac{1}{1+a^2}$, and so the solution for b will still be $b = \overline{Y} - a\overline{X}$. So the new regression line must go through the centroid $\left(\overline{X},\overline{Y}\right)$, in the same way the usual regression line did. Symbolically this is

$$SS=\sum_{i=1}^{n}d_i^2=\sum_{i=1}^{n}\frac{(y_i-ax_i-b)^2}{1+a^2},$$

$$\frac{\partial SS}{\partial b}=\sum_{i=1}^{n}\frac{2\left(y_i-ax_i-b\right)(-1)}{1+a^2}=0,$$

which reduces to

$$\frac{-2}{1+a^2}\sum_{i=1}^{n}(y_i-ax_i-b)=0$$

and then

$$\sum_{i=1}^{n}y_i-a\sum_{i=1}^{n}x_i-nb=0$$

and finally

$$b=\overline{Y}-a\overline{X}.$$

Since this regression also passes through the centroid at $\left(\overline{X},\overline{Y}\right)$, as does usual Least Squares, the liberty is given to simplify the notation considerably. We replace each X_i and Y_i by $X_i-\overline{X}$ and $Y_i-\overline{Y}$, respectively. This makes the centroid $\left(\overline{X},\overline{Y}\right)=(0,0)$. So in what follows all the regression lines will pass through the origin. Also,

$$SS_X=\sum_{i=1}^{n}X_i^2-\frac{\left(\sum\limits_{i=1}^{n}X_i\right)^2}{n}$$

simplifies to

$$SS_X=\sum_{i=1}^{n}X_i^2$$

and

$$SS_Y=\sum_{i=1}^{n}Y_i^2-\frac{\left(\sum\limits_{i=1}^{n}Y_i\right)^2}{n}$$

simplifies to and simplifies to $SS_Y=\sum\limits_{i=1}^{n}Y_i^2$. Also

$$SS_{XY}=\sum_{i=1}^{n}X_iY_i-\frac{\left(\sum\limits_{i=1}^{n}X_i\right)\left(\sum\limits_{i=1}^{n}Y_i\right)}{n}$$

simplifies to

$$SS_{XY} = \sum_{i=1}^{n} X_i Y_i.$$

These standardizations also simplify the residuals for the alternative regression from

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} \frac{(y_i - ax_i - b)^2}{1 + a^2}$$

to

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} \frac{(y_i - ax_i)^2}{1 + a^2}.$$

To find the slope estimate for the alternative equation, differentiate the alternative residual sum of squares with respect to a, set equal to 0, and solve.

$$\sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} \frac{(y_i - ax_i)^2}{1 + a^2}$$

$$\frac{\partial SS}{\partial a} = \sum_{i=1}^{n} \frac{2\left(y_i - ax_i\right)\left(-x_i\right)\left(1 + a^2\right) - 2a\left(y_i - ax_i\right)\left(yli\right) - ax_i}{\left(1 + a^2\right)^2}$$

$$\frac{2}{\left(1 + a^2\right)^2} \sum_{i=1}^{n} \left(-x_i y_i + a x_i^2 - a^2 x_i y_i + a^3 x_i^2 - a y_i^2 + a^2 x_i y_i + a^3 x_i^2\right) = 0$$

$$\begin{aligned} \sum_{i=1}^{n} \left(a^2 x_i y_i + a\left(x_i^2 - y_i^2\right) - xy\right) &= 0 \\ a^2 \sum_{i=1}^{n} x_i y_i + a \sum_{i=1}^{n} \left(x_i^2 - y_i^2\right) - \sum_{i=1}^{n} x_i y_i &= 0 \\ a^2 SS_{XY} + a\left(SS_X - SS_Y\right) - SS_{XY} &= 0 \end{aligned}$$

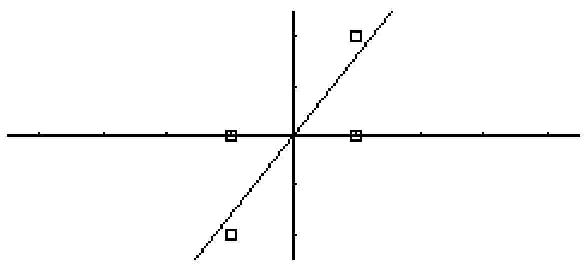
$$a = \frac{\left(SS_Y - SS_X\right) \pm \sqrt{\left(SS_X - SS_Y\right)^2 + 4\left(SS_{XY}\right)^2}}{2SS_{XY}}.$$

Due to the presence of the square root, this equation yields two numbers. One number maximizes the sum of the perpendicular distances and the other minimizes it. The minimizing slope is the slope that yields the best linear regression in comparison to the Method of Least Squares.

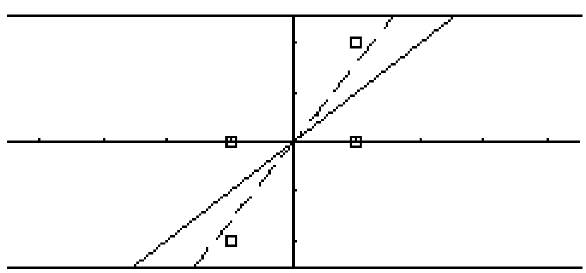
4. Examples

Now we will show some examples of both the Method of Least Squares and the alternative regression.

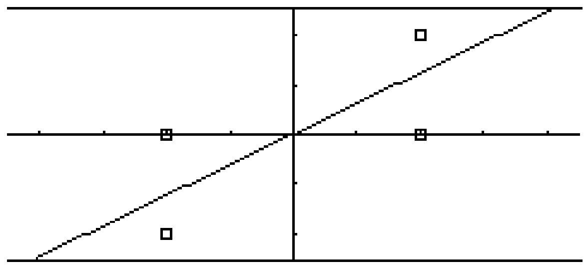
To show the difference between the two regressions, here is an example minimizing the points (1,2), (1,0), (-1,-2), and (-1,0). Holding to our standardizations, and equal zero. By using the quadratic two slopes are returned in the new regression; one that maximizes the perpendicular distance between the points and one that minimizes the perpendicular distance between the points. The maximizing slope however is insignificant and the minimizing slope in this case is 1.618. In this case, the maximum and minimum slopes are negative reciprocals of each other, but this is simply a coincidence and does not often happen. Shown below is the minimizing solution.



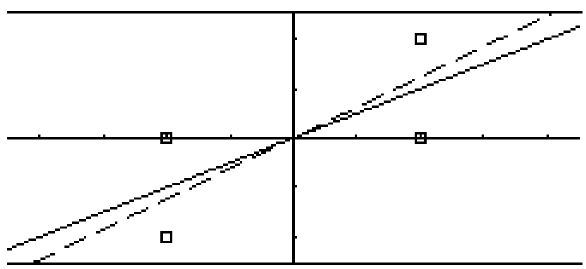
The Method of Least Squares finds the vertical distances between the points. Because the slope for Method of Least Squares is , only and are needed. In this case, the slope is 1. The next graph shows the Method of Least Squares is dashed and the alternative regression is a solid line.



The next example uses the points (2,2), (2,0), (-2,-2), and (-2,0). The minimizing slope for the alternate regression is .618 and the maximizing slope is -1.618. This example is the same as the previous but the ’s are scaled by 2.

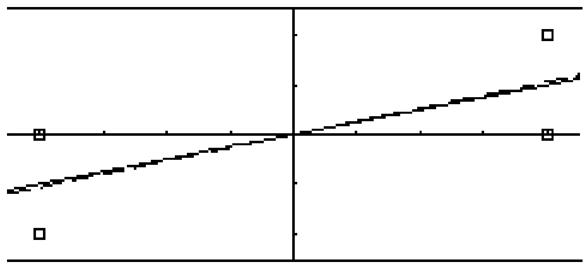


The Method of Least Squares finds a slope of $\frac{1}{2}$ and is shown here dashed and the alternate regression is solid.



However, the alternate regression is scale dependant. By this we mean that the regression slope does not change in a linear manner when one variable is multiplied by a constant. When each was multiplied by 2 , is multiplied by 4, is unchanged, and is multiplied by 2. That makes the usual regression slope change by a factor of 2 from 1 to $\frac{1}{2}$. However, the alternate regression slope changed from 1.618 to 0.618. This would mean that if you used the same people to obtain a regression equation of weight on height, you would predict different weights depending on whether the units were inches and pounds or centimeters and kilograms.

Continuing this, the points (4,2), (4, 0) , (-4, -2) and (-4, 0) come to a slope of , or .2656. The slope for Least Squares is . Shown here are the very similar resulting lines with the alternate regression as a solid line and Least Squares as a dashed line.



Notice that Least Squares finds the same slope as previously for the points (1,2), (1,0), (-1,-2), and (-1,0) scaled by 2 because of the SS_X in the denominator. But the alternate regression found .2656 which is not as easily related to the previous solution (1.618).

Other potential downsides to the alternative method are that it is a more difficult equation and the output results in two possible results. As shown previously, the equation is

$$a = \frac{(SS_Y - SS_X) \pm \sqrt{(SS_X - SS_Y)^2 + 4\,(SS_{XY})^2}}{2SS_{XY}}.$$

This would obviously not be considered easy to calculate. The maximum and minimum outputs are certainly not too difficult to choose between, but it does add another step to the analysis.

While this is not a new concept or original idea, this alternative method is not examined often even though it seems more intuitive. As shown, it is more visually appealing and the regression line seems more consistent with the point pattern. However, the equation leaves much room for mistake and the slope is not easily predicted because it is scalar dependent. In this case as in many times before, this method is not worth unsettling Least Squares as the common method of regression.

Acknowledgements. Special thanks to Dr. Don Tosh, Professor of Mathematics, Evangel University.

The Problem Corner

Edited by Pat Costello

The Problem Corner invites questions of interest to undergraduate students. As a rule, the solution should not demand any tools beyond calculus and linear algebra. Although new problems are preferred, old ones of particular interest or charm are welcome, provided the source is given. Solutions should accompany problems submitted for publication. Solutions of the following new problems should be submitted on separate sheets before January 1, 2010. Solutions received after this will be considered up to the time when copy is prepared for publication. The solutions received will be published in the Spring 2010 issue of The Pentagon. Preference will be given to correct student solutions. Affirmation of student status and school should be included with solutions. New problems and solutions to problems in this issue should be sent to Pat Costello, Department of Mathematics and Statistics, Eastern Kentucky University, 521 Lancaster Avenue, Richmond, KY 40475-3102 (e-mail: pat.costello@eku.edu)

NEW PROBLEMS 641-648

Problem 641. *Proposed by Lisa Kay, Eastern Kentucky University, Richmond, KY.*

Suppose that there are five students enrolled in a chemistry class. They will have to complete five lab assignments. For each lab assignment, four of the students will work in two pairs while one student works independently. Each student will work independently for exactly one of the five labs. Each student will work with each of the other four students exactly once. How many different lab schedules are possible?

Problem 642. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let a, b, c be the lengths of the sides of a triangle ABC with heights h_a, h_b, and h_c, respectively. Prove that

$$\prod_{\text{cyclic}} \left(\frac{h_a}{h_b + h_c} \right)^{1/3} \leq \frac{1}{6} \left(\frac{a+b+c}{\sqrt[3]{abc}} \right).$$

Problem 643. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

The equation $x^3 - 2x^2 - x + 1 = 0$ has three real roots $a > b > c$. Find the value of $ab^2 + bc^2 + ca^2$.

Problem 644. *Proposed by Andrew Cusumano, Great Neck, NY.*

Find two primes whose reciprocals repeat after exactly 7 decimal places.

Problem 645. *Proposed by Ben Thurston, Florida Southern College, Lakeland, FL.*

What is the expected number of rolls of a fair die required to have all six faces come up at least once?

Problem 646. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, Illinois.*

Suppose that n is an odd integer. Show that

$$\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{n-1} - \frac{1}{n} > \frac{1}{4} - \frac{1}{2\,(n+1)}.$$

Problem 647. *Proposed by Panagiote Ligouras, Leonardo da Vinci High School, Noci, Italy.*

Let a, b, and c be the sides, and m_a, m_b, and m_c the medians of a triangle ABC. Prove or disprove that

$$m_a^2 m_b^2 + m_b^2 m_c^2 + m_c^2 m_a^2 \geq \frac{9}{4}\left(\frac{a^4 bc\cos A}{b^2+c^2} + \frac{ab^4 c\cos B}{c^2+a^2} + \frac{abc^4\cos C}{a^2+b^2}\right).$$

Problem 648. *Proposed Ovidiu Furdui, University of Toledo, Toledo, OH.*

Let $k > 1$ be a real number. Find the value of $\int_0^1 \left\{\frac{1}{\sqrt[k]{x}}\right\} dx$, where $\{a\} = a - \lfloor a \rfloor$ denotes the fractional part of a. [For example, $\{1.9\} = 0.9$.]

SOLUTIONS TO PROBLEMS 624-631

Problem 624. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, Illinois.*

Given a tetrahedron, prove that two triangles can be formed such that the lengths of the six triangle sides equal the lengths of the six edges of the tetrahedron. Prove that the converse is not true.

Solution *by the proposers.*

Suppose the six edges of the tetrahedron cannot be arranged to form two triangles. Let the vertices be labeled so that AB is the longest edge of the tetrahedron. If

$$AB < AC + AD,$$

then AB, AC, and AD can be arranged into one triangle and $\triangle BCD$ is a triangular face of the tetrahedron so we would have two triangles. This means that

$$AB \geq AC + AD.$$

Similarly,

$$AB \geq BC + BD.$$

Therefore,

$$2AB \geq AC + AD + BC + BD.$$

However, $\triangle ABC$ is a face of the tetrahedron, so

$$AB < AC + BC.$$

Since $\triangle ABD$ is also a face,

$$AB < AD + BD.$$

Adding these two inequalities gives

$$2AB < AC + BC + AD + BD.$$

This contradicts the previous inequality. Hence the six edges can be arranged to form two triangles.

To prove that the converse is not true, consider the two triangles one being equilateral whose sides have length 1 and the other having sides of length 100, 102, 104. With these lengths, we can form only one triangle with the length 104 edge as one side of the triangle. But each edge of a tetrahedron is an edge of two of the triangles formed by the faces. Therefore, it is not possible to arrange these six edges into a tetrahedron.

Problem 625. *Proposed by Duane Broline and Gregory Galperin (jointly), Eastern Illinois University, Charleston, Illinois.*

All of the integers from 1 through 999999 are written in a row. All of the zeros are erased. Each of the remaining digits is separately inverted and the sum, S, is computed. Let T be the sum of the reciprocals of the digits 1 through 9. Show that S/T is an integer and find it.

Solution *by the proposers.*

Consider the one million six-digit sequences, $xyztuv$, where x, y, z, t, u, and v are between 0 and 9, inclusive. In each of these one million sequences, let each digit be replaced by that digit plus 1 modulo 10 (so that 9 is replaced by 0). The resulting list of sequences is the same, except for order, as the initial list. Thus each digit occurs the same number of times among the one million six-digit sequences. As there are 6,000,000 digits, each of the ten digits occurs 600,000 times. Now suppose all of the zeroes are erased from among the one million six-digit sequences. The digits that remain will be the same as those that would be left if all integers from 1 through 999,999 were written in a row and then all of the zeroes were erased. In each case, each of the digits 1 through 9 occurs 600,000 times. Hence $S = (600,000)T$ and

$$S/T = 600,000.$$

Solution *by Samantha Corvino (student), Slippery Rock University, Slippery Rock, PA.*

Let S_n be the sum of the reciprocals of the nonzero digits of the positive integers less than 10n. Clearly $S_1 = T$. For two-digit numbers, each nonzero digit appears ten times in the tens column and 9 times in the ones column. Thus,

$$S_2 = 10T + 9S_1 + S_1 = 10T + 10S_1.$$

Similarly,

$$S_3 = 100T + 9S_2 + S_2 = 10^2T + 10S_2,$$

and in general

$$S_n = 10^{n-1}T + 10S_{n-1}.$$

One can easily verify that the solution to this difference equation has the form

$$S_n = n10^{n-1}T,$$

from which it can be deduced that $S = S_6 = 600,000T$ and

$$S/T = 600,000.$$

Also solved by Carl Libis, University of Rhode Island, Kingston, RI, Taylor Franzman (student), California State University-Fresno, Fresno, CA, Erik Murphy (student), Waynesburg University, Waynesburg, PA, Parker Richey (student), Northeastern Oklahoma State University, Tahlequah, OK.

Problem 626. *Proposed by David Rose, Florida Southern College, Lakeland, FL.*

Two values are randomly selected from the uniform distribution on the interval $(0, L)$. They create three subintervals of the interval $[0, L]$. What is the probability that the lengths of the three subintervals are the lengths of the sides of some triangle?

Solution *by Chip Curtis, Missouri Southern State University, Joplin, MO.*

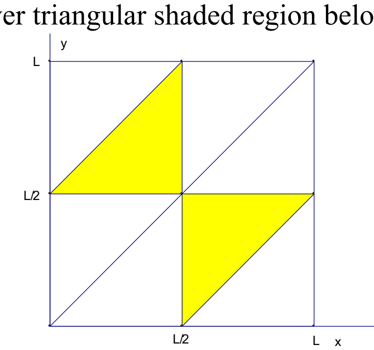
Regard the two randomly-selected values as coordinates of an ordered pair (x, y) in the xy-plane. The feasible region is the square with vertices $(0, 0)$, $(L, 0)$, $(0, L)$, and (L, L), together with its interior.

Case 1 Suppose $0 < x \leq y < L$. In this case, the three segment lengths are x, $y - x$, and $L - y$. These lengths will be side lengths for a triangle if and only if they satisfy all three inequalities required by the triangle inequality:

$$\begin{aligned} L - y &< x + (y - x) = y \Longleftrightarrow y > L/2 \\ y - x &< L - y + x \Longleftrightarrow y < L/2 + x \\ x &< L - y + y - x = L - x \Longleftrightarrow x < L/2 \end{aligned}$$

The solution to this system of inequalities corresponds to the upper triangular shaded region below.

Case 2 Suppose $0 < y < x < L$. The segment lengths are y, $x - y$, and $L - x$, with those corresponding to sides of a triangle when the x and y values are in the lower triangular shaded region below.



Since the total shaded area represents $\frac{1}{4}$ of the total area of the square, the probability that the lengths constructed are the sides of a triangle is $\frac{1}{4}$.

Also solved by Lisa Kay, Eastern Kentucky University, Richmond, KY, Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, and the proposer.

Problem 627. *Proposed by Ken Dutch, Eastern Kentucky University, Richmond, KY.*

Suppose that the artist Krypto wants to form several rows of blocks 10 feet wide. He only wants to use two types of blocks - one type is one foot wide and the other is two feet wide. He wants to form a row for every possible pattern of blocks (order matters). How many rows will he have to make? How many of each type of block will he have to use?

Solution *by Alycia Butchelli and Karissa King (students), Slippery Rock University, Slippery Rock, PA.*

There are only six basic combinations of the blocks as described below. Each must be permuted. Given that there are repeated elements in each combination, we obtain:

	m	d	P
□ □ □ □ □ □ □ □ □ □	10	0	$\frac{10!}{10!} = 1$
□ □ □ □ □ □ □ □ ▭	8	1	$\frac{9!}{8!1!} = 9$
□ □ □ □ □ □ ▭ ▭	6	2	$\frac{8!}{6!2!} = 28$
□ □ □ □ ▭ ▭ ▭	4	3	$\frac{7!}{4!3!} = 35$
□ □ ▭ ▭ ▭ ▭	2	4	$\frac{6!}{2!4!} = 15$
▭ ▭ ▭ ▭ ▭	0	5	$\frac{5!}{5!} = 1$

,

where m is the number of monominoes, d is the number of dominoes, and P is the number of permutations.

There are 89 permutations, so there are 89 rows needed. To find the number of each individual block type being used, multiply the number of arrangements for each combination by the number of each type of block in the combination and add. The number of monominoes used is 420. The number of dominoes used is 235.

Also solved by Carl Libis, University of Rhode Island, Kingston, RI and the proposer.

Problem 628. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let F_n be the n^{th} Fibonacci number, defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Prove that

$$\left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} F_k \tanh F_k\right)^2 + \left(\frac{1}{\sqrt{n}}\sum_{k=1}^{n} F_k \operatorname{sech} F_k\right)^2 \leq F_n F_{n+1}.$$

Solution *by the proposer.*

Applying the Cauchy-Schwarz inequality to the vectors $\overrightarrow{u} = (F_1, F_2, ..., F_n)$ and $\overrightarrow{v} = (\tanh F_1, \tanh F_2, ..., \tanh F_n)$ and taking into account that

$$F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1},$$

as can be easily proved, we get

$$\begin{aligned}\left(\sum_{k=1}^{n} F_k \tanh F_k\right)^2 &\leq \left(F_1^2 + F_2^2 + \cdots + F_n^2\right)\left(\sum_{k=1}^{n} \tanh^2 F_k\right)\\ &= F_n F_{n+1}\left(\sum_{k=1}^{n} \tanh^2 F_k\right).\end{aligned}$$

Applying the Cauchy-Schwarz inequality to the vectors $\overrightarrow{u} = (F_1, F_2, ..., F_n)$ and $\overrightarrow{v} = (\operatorname{sech} F_1, \operatorname{sech} F_2, ... \operatorname{sech} F_n)$, we have

$$\begin{aligned}\left(\sum_{k=1}^{n} F_k \operatorname{sech} F_k\right)^2 &\leq \left(F_1^2 + F_2^2 + \cdots + F_n^2\right)\left(\sum_{k=1}^{n} \operatorname{sech}^2 F_k\right)\\ &= F_n F_{n+1}\left(\sum_{k=1}^{n} \operatorname{sech}^2 F_k\right).\end{aligned}$$

Since $\tanh^2 F_k + \operatorname{sech}^2 F_k = 1$ (in general, $\tanh^2 x + \operatorname{sech}^2 x = 1$), we obtain, after adding the previous expressions,

$$\left(\sum_{k=1}^{n} F_k \tanh F_k\right)^2 + \left(\sum_{k=1}^{n} F_k \operatorname{sech} F_k\right)^2 \leq n F_n F_{n+1}$$

from which the desired conclusion follows. Equality holds when $n = 1$.

Problem 629. *Proposed by Jose Luis Diaz-Barrero, Universitat Politecnica de Catalunya, Barcelona, Spain.*

Let a, b, c be real numbers, with $a, b, c \geq 1$. Prove that

$$\frac{a^{1/a}}{b^{1/b}+c^{1/c}}+\frac{b^{1/b}}{a^{1/a}+c^{1/c}}+\frac{c^{1/c}}{b^{1/b}+a^{1/a}}<2.$$

Solution *by the proposer.*

First, we will see that with segments of lengths $a^{1/a}$, $b^{1/b}$, $c^{1/c}$, it is always possible to build a triangle. In fact, we have $[a] \leq a < a+1$. Applying Bernoulli's inequality yields $2^a \geq 2^{[a]} = (1+1)^{[a]} \geq 1+[a] > a \geq 1$ from which we get $2 > a^{1/a} \geq 1$. Likewise $2 > b^{1/b} \geq 1$ and $2 > c^{1/c} \geq 1$ and

$$\begin{aligned} a^{1/a}+b^{1/b} &\geq 1+1=2>c^{1/c} \\ b^{1/b}+c^{1/c} &\geq 1+1=2>a^{1/c} \\ c^{1/c}+a^{1/a} &\geq 1+1=2>b^{1/b}. \end{aligned}$$

Therefore it is always possible to build a triangle with the lengths $a^{1/a}$, $b^{1/b}$, $c^{1/c}$. Now we have

$$\begin{aligned} a^{1/a}+b^{1/b} &> \left(a^{1/a}+b^{1/b}+c^{1/c}\right)/2 \\ b^{1/b}+c^{1/c} &\geq \left(a^{1/a}+b^{1/b}+c^{1/c}\right)/2 \\ c^{1/c}+a^{1/a} &\geq \left(a^{1/a}+b^{1/b}+c^{1/c}\right)/2. \end{aligned}$$

Inverting the preceding inequalities, and multiplying by $c^{1/c}$, $a^{1/a}$, $b^{1/b}$, respectively, we obtain

$$\begin{aligned} \frac{c^{1/c}}{a^{1/a}+b^{1/b}} &< \frac{2c^{1/c}}{a^{1/a}+b^{1/b}+c^{1/c}} \\ \frac{a^{1/a}}{c^{1/c}+b^{1/b}} &< \frac{2a^{1/a}}{a^{1/a}+b^{1/b}+c^{1/c}} \\ \frac{b^{1/b}}{a^{1/a}+b^{1/b}} &< \frac{2b^{1/b}}{a^{1/a}+b^{1/b}+c^{1/c}}. \end{aligned}$$

Adding these inequalities and rearranging terms gives the desired conclusion.

Problem 630. *Proposed by the editor.*

Suppose that $\log_x y + \log_y x$ is a positive integer. Prove that $(\log_x y)^n + \left(\log_y x\right)^n$ is an integer for all positive integers n.

Solution *by Daniel Parkes and Kevin Rose (students), California State University-Fresno, Fresno, CA.*

Let $s = \log_y x$ and $t = \log_x y$. Then $y^s = x$ and $x^t = y$. By substitution, $\left(x^t\right)^s = y^s = x$, so $x^{st} = x^1$. Equating exponents, $st = 1$, so that s and t are inverses. We proceed by induction. The smallest case is $n = 1$ which is the same as the original assumption. Assume that $s^k + t^k$ is an integer for all k with $1 \leq k \leq n$. Consider the case of $s^{n+1} + t^{n+1}$. We break into two cases:

Case 1 $n + 1$ even, so $n + 1 = 2u$ for some integer u. Then

$(s+t)^{n+1} = s^{n+1} + a_1 s^n t + a_2 s^{n-1} t^2 + \cdots + a_u s^u t^u + \cdots + a_1 s t^n + t^{n+1}$,

with the a_i being binomial coefficients. Since $st = 1$, we have

$$\begin{aligned} (s+t)^{n+1} &= s^{n+1} + a_1 s^{n-1} + a_2 s^{n-3} + \cdots + a_u + \cdots \\ &\quad + a_1 t^{n-1} + t^{n+1} \\ &= s^{n+1} + a_1\left(s^{n-1} + t^{n-1}\right) + a_2\left(s^{n-3} + t^{n-3}\right) + \cdots \\ &\quad + a_u + t^{n+1}. \end{aligned}$$

By the induction hypothesis, $s^{n-1} + t^{n-1}$, $s^{n-3} + t^{n-3}$, ... are all integers. Since all of the binomial coefficients are integers and $(s+t)^{n+1}$ is an integer, $s^{n+1} + t^{n+1}$ is an integer.

Case 2 $n + 1$ odd, so $n = 2u$ for some integer u. Then

$$\begin{aligned} (s+t)^{n+1} &= s^{n+1} + a_1 s^n t + a_2 s^{n-1} t^2 + \cdots + a_u s^{u+1} t^u + a_u s^u t^{u+1} \\ &\quad + \cdots + a_1 s t^n + t^{n+1} \\ &= s^{n+1} + a_1 s^{n-1} + a_2 s^{n-3} + \cdots + a_u s + a_u t \\ &\quad + \cdots + a_1 t^{n-1} + t^{n+1} \\ &= s^{n+1} + a_1\left(s^{n-1} + t^{n-1}\right) + a_2\left(s^{n-3} + t^{n-3}\right) \\ &\quad + \cdots + a_u\left(s+t\right) + t^{n+1}. \end{aligned}$$

As in case 1, all of the middle terms on the right are integers, and the left side is an integer, so $s^{n+1} + t^{n+1}$ is an integer.

Thus by mathematical induction, $(\log_x y)^n + \left(\log_y x\right)^n$ is an integer for all positive integers n.

Also solved by Joan Bell, Northeastern Oklahoma State University, Tahlequah, OK, Russell Euler and Jawad Sadek, Northwest Missouri State University, Maryville, MO, and the proposer.

Problem 631. *Proposed by the editor.*

The Columbus State University Problem of the Week for March 10, 2008 asked for the three smallest positive integers that could not be written as the difference of two positive prime numbers. These turn out to be primes. Prove that there are infinitely many positive primes that cannot be written as the difference of two positive prime integers. Also prove that there are infinitely many pairs of positive integers $(n, n+2)$ that cannot be written as the difference of two positive primes.

Solution *by the proposer.*

Let p be a prime with $p \equiv 3 \pmod{10}$. Then p is an odd prime. If p can be written as the difference of two positive primes, say $p = q - r$, then $q > p$ and so q is odd. Since p and q are odd, r would need to be even. The only even prime is 2 which forces $q = p + 2 \equiv 5 \pmod{10}$ which is impossible if q is prime. Hence p cannot be written as the difference of two primes. By Dirichlet's Theorem on Primes in Arithmetic Progressions we know there are infinitely many primes congruent to 3 $(\text{mod}\, 10)$ and none of these can be written as the difference of two primes.

For the second part of the problem, let n be a prime with $n \equiv 3$ $(\text{mod}\, 70)$. So n is odd. If n can be written as the difference of two positive primes, say $n = q - r$, then q would be odd, $r = 2$, and $q = n + 2 \equiv 5$ $(\text{mod}\, 70)$ which is impossible for a prime q. Hence n cannot be written as the difference of two primes. In addition, if $n + 2$ can be written as the difference of two primes, say $q - r$, then q would be odd, $r = 2$, and $q = (n + 2) + 2 \equiv 7 \pmod{70}$ which is also impossible. So neither n nor $n + 2$ can be written as the difference of two primes. Since 3 is relatively prime to 70, Dirichlet's Theorem on Primes in Arithmetic Progressions guarantees that there are infinitely many primes $n \equiv 3 \pmod{70}$. So there are infinitely many pairs of positive integers $(n, n+2)$ that cannot be written as the difference of two positive primes.

Kappa Mu Epsilon News

Edited by Connie Schrock, Historian

Updated information as of March 2009

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Installation Report

New York Pi
Mount Saint Mary College, Newburgh, NY

The installation of the New York Pi Chapter of Kappa Mu Epsilon was held on Thursday evening, 29 March 2007, in the Villa Library of Mount Saint Mary College in Newburgh, New York. Dr. Andrew M. Rockett, a former editor of The Pentagon, was the installing officer. A welcoming address was delivered by Dr. Iris Turkenkopf, Vice-President for Academic Affairs, and a mathematical talk on "A Continuing Experience" was presented by Amy LaPointe before the fourteen charter members were initiated and the new chapter installed. The new members received their membership certificates and KME pins during the ceremony and will receive their honor cords for graduation at a college honor society convocation later this spring.

Officers installed were: Kelly A. Reid, president; Amy LaPointe, vice-president; Brianne Beebe, recording secretary; Erica Lauffer, treasurer; and Dr. Lee Fothergill, faculty sponsor and corresponding secretary.

The evening began with a faculty dinner hosted by Sister Patricia Sullivan, department chair, and Dr. Lee Fothergill, and concluded with a reception and refreshments for the initiates, faculty, friends, and parents.

Chapter News

AL Gamma – Montevallo University

John Herron, Corresponding Secretary.

New Initiates – Claire E. Carter, David E. Skelton, Brittany Correia.

HI Alpha – Hawaii Pacific University

Ronnie Crane, Corresponding Secretary.

New Initiates – Syed Iskandar Alsagoff, Gena Goodson, Guy Higuchi.

IA Alpha – University of Northern Iowa

Chapter President – Kellen Miller, 12 Current Members, 2 New Members

Darcy Thomas, Vice–President; Megan Klein, Secretary; Beth Kolsrud, Treasurer; Mark D. Ecker, Corresponding Secretary.

Our first Fall KME meeting was held on September 25, 2008 at Professor Jerry Ridenhour's house where student member Darcy Thomas presented her paper entitled "Analysis of Select Iowa Towns". The University of Northern Iowa Homecoming Coffee was held at Professor Suzanne Riehl's residence on October 11, 2008. Student member Michelle Gogerty presented her paper entitled "Growth Among Iowa Counties" at our second meeting on November 13, 2008 at Professor Mark Ecker's home. Student member Megan Klein addressed the fall initiation banquet with "Analysis of Domestic Gross Revenue for Feature Films". Our Fall banquet was held at the Brown Bottle restaurant in Cedar Falls on December 11, 2008 where two new members were initiated.

New Initiates – Brendt Baedke, Kelsey Staudacher.

IL Theta – Benedictine University

Chapter President – Nicholas Meyer, 10 Current Members, 4 New Members

Other fall 2008 officer: Manmohan Kaur, Corresponding Secretary.

IL Zeta – Dominican University

Chapter President – Nancy Gullo and Monika Vidmar, 20 Current Members, 10 New Members

Phillip Lenzini, Secretary; Angelina Myers, Treasurer; Aliza Steurer, Corresponding Secretary.

Dr. Paul Coe (chair of the math department) gave a talk on how to make platonic solids out of origami. We held four monthly meetings and three officers' meetings.

IN Beta – Butler University

Chapter President– Brent Freed, 14 Current Members, 5 New Members

Marcus Such, Vice–President; Keenan Hecht, Secretary; Jessica Bowman, Treasurer; Amos Carpenter, Corresponding Secretary.

In addition to our monthly meetings we brought two invited speakers to campus. Dr. Alain Togbe, Associate Professor of Mathematics at Purdue University North Central, Westville, Indiana, presented On the Family of Diophantine Triples. Dr. David Housman, Professor of Mathematics at Goshen College, Goshen, Indiana, presented Strategic and Cooperative Games.

New Initiates – Samantha Bane, Jessica Bowman, Caitlin Mallon, Timothy Maurer, Elizabeth Wiley.

IN Delta – University of Evansville

Chapter President – Jackie Rice, 38 Current Members, 0 New Members

McLane Crowell, Vice-President; Danny Price, Secretary; Dr. Adam Salminen, Correspond Secretary.

KS Alpha – Pittsburg State University

Dr. Tim Flood, Corresponding Secretary.

New Initiates – Daniele Cunningham, Ashley Curran, Seang-Hwane Joo, Emily Krysztof, Jennifer McCracken, Keenan Meeker, Robert Merrill, Aaron Poe, Ryan Sorell, Kevin Spencer, Shauna Wachter, Penghua, Katlyn Leslie.

KS Beta – Emporia State University

Chapter President– Whitni Turley, 26 Current Members, 5 New Members

Melissa Swager, Vice–President; Ryan Wilson, Secretary; Mirel Howard, Treasurer; Connie Schrock, Corresponding Secretary.

KS Delta –Washburn University

Chapter President– Brandy Mann, 30 Current Members, 10 New Members

Jackson Waechter, Vice–President; Sarah Butler, Secretary; Sarah Butler, Treasurer; Mike Mosier, Corresponding Secretary.

The Kansas Delta chapter of KME met for four luncheon meetings with the Washburn Math Club during the semester. Our chapter's annual initiation banquet and ceremony will be held on February 23, 2009. Two students, Christine Potter and Sarah Butler, are planning to present papers at the KME National Convention in Philadelphia, PA in April of 2009.

KS Gamma – Benedictine College

Chapter President – Matthew Weaver, 8 Current Members, 0 New Members

Caitlin Kelly, Vice-President; Christina Henning, Secretary; Christina Henning, Treasurer; Dr. Linda Herndom, Corresponding Secretary.

Our chapter took an active part in arranging the "Back to School" cookout for students interested in mathematics and computer science on September 17 at Jackson Park in Atchison, Kansas. A large group of students enjoyed hamburgers and hot dogs on a beautiful fall day.

KY Beta – University of the Cumberlands

Chapter President- Teresa Shafer, 29 Current Members, 0 New Members
Dustin Ursrey, Vice–President; Joshua Ward, Secretary; Andrzej Lenard, Treasurer; Dr. Jonathan Ramey, Corresponding Secretary.

Along with the Mathematics and Physics Club and Sigma Pi Sigma, the chapter had a picnic at Briar Creek Park on October 9. On December 12, the entire department, including the Math and Physics Club, the Kentucky Beta chapter, and Sigma Pi Sigma had a Christmas party with about 34 people in attendance.

MD Alpha – College of Notre Dame of Maryland

Chapter President – Nicole Kotulak, 11 Current Members, 12 New Members
Shannon Kramer, Vice-President; Andrea Weedn, Secretary; Kristen Clarke, Treasurer; Dr. Margaret Sullivan, Corresponding Secretary.

Our November Induction included a presentation by Dr.Joseph Ganem of Loyola College in Baltimore on the mathematics of economics based on his book: *The Two-Headed Quarter*.

MD Beta – McDaniel College

Chapter President – Shannon Jackson, Current Members, New Members
Kristin Duling, Vice-President; Stephen Hardy, Secretary; Wesley Mann, Treasurer; Erin Ralsamo, Historian; Dr. Harry Rosenzweig, Corresponding Secretary.

The McDaniel College chapter was extremely active this past semester. They ran weekly tutoring sessions for all of our freshman and sophomore level courses, and sponsored a talk on "Does the Klein Bottle hold Water?" at their fall induction ceremony. In addition, they sponsored a fall picnic for all current and prospective mathematics majors, met with the KME chapters from Hood College and Mount Saint Mary's at a picnic at Mount Saint Mary's, had a bowling match against the physics majors here, and sponsored a free game night at the student center.

New Initiates – David Grimes, Fenghao Wang.

MD Delta – Frostburg State University

Chapter President – Lisa Gitelman, 15 Current Members, 0 New Members.
Kelly Seaton, Vice-President; Kelly Seaton, Secretary; Joseph Bascelli, Treasurer; Dr. Mark Hughes, Corresponding Secretary.

The Maryland Delta Chapter started the semester with a meeting in September where we planned our participation in a "majors fair" held in the student center. This event was held in order to introduce new students to the various majors and student organizations present on campus. Our members represented the Department of Mathematics and KME. Displays and multimedia presentations were prepared and the fair went very nicely.

During our October meeting, we watched some interesting animations on how to visualize four dimensional objects. Dr. Frank Barnet provided our activity for the November meeting with an amazing presentation on using POV Ray software to create mathematical animations, including a double pendulum as an illustration of chaotic behavior.

MD Epsilon – Stevenson University

Chapter President – Krystal Burns, 18 Current Members, 7 New Members

Catherine Gerber, Vice-President; Liesl Feinour, Secretary; Brandon Cooper, Treasurer; Dr. Christopher E. Barat, Corresponding Secretary.

On September 10, 2008, seven new members were initiated into the MD Epsilon chapter. The guest speaker, Dr. Michael O'Leary of Towson University, spoke on "Using Mathematics to Catch Criminals." The Fall Raffle raised over $1200 for the chapter and some of the proceeds will be used to send students to local mathematics meetings, including the KME biennial convention in March. Planned Spring activities include the second annual "pie sale" in honor of Pi Day and an afternoon of mathematical presentations to commemorate Mathematics Awareness Month in April.

New Initiates – Justin Bobo, Lauren Bonsal, Liesl Feinour, Jonathon Kincer, Jennifer Kurek, Alyssa Pawlowicz, Timothy Potter.

MI Beta – Central Michigan University

Chapter President – Stacy Blemaster, 10 Current Members, 2 New Members

Jeremy Thomson, Vice-President; Gerald Haynes, Secretary; Kim Paweleski, Treasurer; Sivaram K. Narayan, Corresponding Secretary.

During fall 2008 Kappa Mu Epsilon (KME) met every other Wednesday evening. Members raised money through a book sale held jointly with other student organizations in the department. The money raised from the book sale and membership dues was used for buying pizza on meeting days and for conducting induction ceremony. During fall 08 KME members started raising money to attend the KME National Convention in Philadelphia in March 2009. They sold discount coupons for local restaurants. Two new members were inducted in fall 08. Dr. Arnie Hammel from CMU gave a talk on October 27, 2008 on "Symmetric-Key and Public-Key Cryptosystems". Dr. Sid Graham was the keynote speaker in the fall 08 induction ceremony held on November 16, 2008. He spoke on "Triangular Peg Solitaire".

MO Alpha – Missouri State University

Chapter President– Chris Inabnit, 31 Current Members, 10 New Members
Michael McDonald, Vice–President; Nicole Jones, Secretary; Bobbi Gregory, Treasurer; Jorge Rebaza, Corresponding Secretary.

09/15/08	Seminar	Kishor Shah, MSU
09/22/08	Annual Picnic	
10/27/08	Seminar	Matthew Wright, MSU
11/20//08	Seminar	Nicole Typaldos, Bobbi Gregory, MSU

New Initiates – Meagan Bunge, Jeff Chapman, Antoinette Dunn, Ashley King, Patrick Margavio, Stephen Parry, Alicia Russell, Jacob Swett, Ryan Thomas, Jordan Wilson.

MO Beta – Central Missouri State University

Chapter President – Thomas Gossell, 25 Current Members, 6 New Members
Todd Carlstrom, Vice-President; Phat Hoang, Secretary; Cynthia Craft, Treasurer; Dr. Rhonda McKee, Corresponding Secretary.

New Initiates – Cynthia Craft, Ian Ewing, William Hall, Paul Kelly, Abdoulaye Singuy Ndong, Julia Knapp.

MO Iota – Missouri Southern State University

Chip Curtis, Corresponding Secretary.

MO Nu – Columbia College

Chapter President – Kristin Crane, 12 Current Members, 0 New Members
Megda Pride, Vice-President; Andrew Grote, Secretary; Neal Lines, Treasurer; Dr. Ann Bledsoe, Corresponding Secretary.

KME sponsored a Calculus III applied projects presentation event held on Dec. 2, 2008;
also KME participated in the Columbia College Family Day celebration.

MO Theta – Evangel University

Chapter President – Adrienne Arner, 8 Current Members, 0 New Members
Rebekah Holmes, Vice-President; Don Tosh, Corresponding Secretary.

Meetings were held monthly. In December we had our semester social at the home of Don Tosh.

MS Alpha – Mississippi University for Women

Chapter President – Dana Derrick, 12 Current Members, 1 New Member
Stefanie Zegowitz, Vice-President; Stefanie Zegowitz, Secretary; Dana Derrick , Treasurer; Dr. Shaochen Yang, Corresponding Secretary.

The Mississippi Alpha Chapter's initation ceremony was held on Thursday, September 18, 2008. Several KME faculty and student members presented sessions at the Sonya Kovalevsky High School Mathematics Day held on Mississippi University for Women campus on October 7, 2008. During our November meeting, we prepared several shoe boxed for

"Operation Christmas Child" of Samaritan's Purse which helps deprived children.

New Initiate – LaBeausha Holt.

NE Beta – University of Nebraska at Kearney

Current President – Corey Hatt, 12 Current Members, 2 New Members
Drew Rische, Vice-President; Robert Langan, Secretary; Sasha Andersen, Treasurer; Dr. Katherine Kime, Corresponding Secretary.

In October, 2009, KME and the Dept. of Mathematics and Statistics sponsored an Open House, in which students could meet faculty and other students. We played Set, Jinga, dominoes, and created interesting tilings with the magnetic tiles activity, Fractiles-7. We received $300 from student government for food for the event. Our Vice-President, Drew Rische, successfully presented our application for the funds. A KME member, Cameron Push, designed a database for KME to use for recording initiations, minutes of meetings, etc., as part of a computer science course he was taking.

NY Omicron – St. Joseph's College

Chapter President – Heather Kramer, 25 Current Members, 0 New Members
Nicole Hatzispirou, Vice–President; Kristine Vaccaro, Secretary; Brian Callen, Treasurer; Elana Epstein, Corresponding Secretary.

In Fall 2008 our chapter held some fundraisers, including bagel sales and bake sales. All of our members volunteered to tutor at our Saturday morning math clinic for local high school students.

OH Epsilon – Marietta College

Chapter President – Kelsie McCartney, 25 Current Members, 0 New Members
Megan Brothers, Vice-President; Dr. John C. Tynan, Corresponding Secretary.

OK Alpha – Northeastern State University

Chapter President – Caleb Knowlton, 60 Current Members, 12 New Members
Aaron LeBounty, Vice–President; Callie Wilson, Secretary; Angela Arnold, Treasurer; Dr. Joan E. Bell, Corresponding Secretary.

Our fall initiation brought 12 new members into our chapter. We sponsored several speakers this semester. Dr. Alberto Striolo, University of Oklahoma, spoke on "Molecular Simulations and Computer Graphics: Modern Tools for Chemical Engineering Research." Peggy Hladik and Misty Megee gave a presentation on the Oklahoma Residency Program for new teachers. Dr. Deborah Carment and Chuck Pack gave us some hands-on experience with a SMART board.

We ended the semester with a Christmas party for KME members, math majors, and faculty. Dr. Darryl Linde, our department chair, treated us to his famous homemade pizza.

New Initiates – Patricia Bailey, Lori Bottger, Megan Brown, James Bryant, Brandon Childress, Stephanie Emerson, Daniel Hayman, Kristi McCabe, Cameron Richardson, Toni Slagle, Kaiti Smith, Katherine Thompson.

OK Delta – Oral Roberts University

Chapter President – Nathan Marth, 177 Current Members, 7 New Members

Tyler Todd, Vice-President; Jacob Garner, Secretary; Jacob Garner, Treasurer; Vincent E. Dimiceli, Corresponding Secretary.

OK Gamma – Southwest Oklahoma State University

Bill Sticka, Corresponding Secretary.

New Initiates – Amy Cain, Kurtis Eckhardt, Bizuayehu Kebede, Clayton Lindsey, Keturah Odoi, Ahwan Pandey.

PA Lambda – Bloomsburg University of Pennsylvina

Dr. Elizabeth Mauch, Corresponding Secretary.

New Initiates – Caryn Borascius, Shari Jenkins, Joseph Perott, Cori Rubel, Michael Sosnoski, Kyle Von Blohn.

PA Mu – Saint Francis University

Chapter President – Tim Gaborek, 30 Current Members, 0 New Members

Kurt Hoffman, Vice – President; Abi May, Secretary; Kaitlyn Snyder, Treasurer; Dr. Peter Skoner, Corresponding Secretary.

The Pennsylvania Mu Chapter elected officers for the 2008-2009 academic year, choosing senior mathematics/computer science major Tim Gaborek as president, junior mathematics major Kurt Hoffman as vice president, senior mathematics/education major Abigale May as secretary, and senior mathematics/education major Kaitlyn Snyder as treasurer.

As part of the eighth annual Saint Francis University Day of Reflection on October 28, 2009, several KME members participated in a road-cleanup. Fourteen students, along with corresponding secretary Peter Skoner, picked 13 bags of litter along Manor Drive, as part of the Pennsylvania Adopt-a-Highway program.

Several KME faculty and student members participated in the Fifteenth Annual Science Day held on campus on November 25, 2008. Some KME members served as session moderators for faculty making presentations. Others served as moderators, judges, scorekeepers, and timers for the Science Bowl. A total of 437 high school students from 25 high schools attended.

PA Nu – Ursinus College

Jeffrey Neslen, Corresponding Secretary.

New Initiates – Nadine Burtt, David Darmon, Jennifer Heavener, Sarah Hurtt, Scott Kulp, Richard Veale.

PA Sigma– Lycoming College

Chapter President – Laura Silks, 9 Current Members, 0 New Members
David Yanick, Vice-President; B.J. McFadden, Secretary; Christopher Dahlheimer, Treasurer; Dr Santu de Silva, Corresponding Secretary.

TN Delta – Carson-Newman College

Chapter President – Ben Love, 17 Current Members, 5 New Members
Gretchen Hill, Vice-President; Jacob Allen, Secretary; B. A. Starnes, Corresponding Secretary.

The Tennessee Delta Chapter had an active Fall 2008 semester. We had our annual fall picnic at the TVA Cherokee Dam in October. The picnic was highlighted by new member initiation, a talk given by former student/current HS teacher Brian McLaughlin, and a 7-6 student victory in the faculty-student Lacrosse game. In November the chapter held its annual game night. Pizza was served and much fun was to be had playing “Apples to Apples” and “Strat-O-Matic College Football” ... where the faculty did win the Herring Bowl.

TN Epsilon – Bethel College

Chapter President – Jessica Smith, 10 Current Members, 0 New Members
Justin Dubruiel, Vice-President; William Robertson, Secretary; Randon Prather, Treasurer; Mr. Russell Holder, Corresponding Secretary.

TN Gamma – Union University

Chapter President – Kristen Kirk. Current Members, New Members
William Sipes, Vice–President; Sarah Conway, Secretary; Sarah Conway, Treasurer; Kent Willis, Webmaster; Kent Willis, Historian; Bryan Dawson, Corresponding Secretary.

The TN Gamma chapter began its fall activities with an on-campus cookout October 2, complete with hamburgers, hotdogs and an appearance by the Great Dawsoni. We also held a Christmas potluck and “Dirty Santa” gift exchange at the home of Dr. Lunsford on December 4.

TX Gamma – Texas Woman’s University

Dr. Mark Hamner, Corresponding Secretary.

New Initiates – Yevgeny Armor, Rebeka Bennett, Heather Peters, Lindsey Crumpler, Megan Sell, Julie Mooneyham, Gin Gia Wen, Abby Peters, Juana Morales.

TX Iota – McMurry University

Dr. Kelly L. McCoun, Corresponding Secretary.

New initiates – Robert Castaneda, Patricia Cook, Heather Dills, Stephanie Graham, Chrystan Hankins, Becky Hoffman, Zach Leverton, Kevin Leverton, Kevin Morris, Chris Santos, Jeanette Schofield, Clay Schubert, George Valdez, Vic Vaughan, Brent Voorhees.

VA Beta – Radford University

Neil Sigmon, Corresponding Secretary.

New Initiates – Laura Sweat, Kiera Poplawski, Tracy Frazier, Gabby Ness, Kayla Love, Jaclyn Rose, Patterson Rogers.

WI Gamma – University of Wisconsin-Eau Claire

Dr. Simei Tong, Corresponding Secretary.

New Initiates – Chelsey Drohman, Sam Fortuna, Ryan D. Frank, Garrick Fults, Paige Hehl, Kaitlyn Hellenbrand, Alexander C. Larsen, Katie Nestingen, Nicholas Pairolero, Kelli Radloff, Jason Schlough, Erin Stuckert.

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KME National Website:
http://www.kappamuepsilon.org/

Active Chapters of Kappa Mu Epsilon

Listed by date of installation

Chapter	Location	Installation Date
OK Alpha	Northeastern State University, Tahlequah	18 April 1931
IA Alpha	University of Northern Iowa, Cedar Falls	27 May 1931
KS Alpha	Pittsburg State University, Pittsburg	30 Jan 1932
MO Alpha	Missouri State University, Springfield	20 May 1932
MS Alpha	Mississippi University for Women, Columbus	30 May 1932
MS Beta	Mississippi State University, Mississippi State	14 Dec 1932
NE Alpha	Wayne State College, Wayne	17 Jan 1933
KS Beta	Emporia State University, Emporia	12 May 1934
AL Alpha	Athens State University, Athens	5 March 1935
NM Alpha	University of New Mexico, Albuquerque	28 March 1935
IL Beta	Eastern Illinois University, Charleston	11 April 1935
AL Beta	University of North Alabama, Florence	20 May 1935
AL Gamma	University of Montevallo, Montevallo	24 April 1937
OH Alpha	Bowling Green State University, Bowling Green	24 April 1937
MI Alpha	Albion College, Albion	29 May 1937
MO Beta	University of Central Missouri, Warrensburg	10 June 1938
TX Alpha	Texas Tech University, Lubbock	10 May 1940
KS Gamma	Benedictine College, Atchison	26 May 1940
IA Beta	Drake University, Des Moines	27 May 1940
TN Alpha	Tennessee Technological University, Cookeville	5 June 1941
MI Beta	Central Michigan University, Mount Pleasant	25 April 1942
NJ Beta	Montclair State University, Upper Montclair	21 April 1944
IL Delta	University of St. Francis, Joliet	21 May 1945
KS Delta	Washburn University, Topeka	29 March 1947
MO Gamma	William Jewell College, Liberty	7 May 1947
TX Gamma	Texas Woman's University, Denton	7 May 1947
WI Alpha	Mount Mary College, Milwaukee	11 May 1947
OH Gamma	Baldwin-Wallace College, Berea	6 June 1947
CO Alpha	Colorado State University, Fort Collins	16 May 1948
MO Epsilon	Central Methodist College, Fayette	18 May 1949
MS Gamma	University of Southern Mississippi, Hattiesburg	21 May 1949
IN Alpha	Manchester College, North Manchester	16 May 1950
PA Alpha	Westminster College, New Wilmington	17 May 1950
IN Beta	Butler University, Indianapolis	16 May 1952
KS Epsilon	Fort Hays State University, Hays	6 Dec 1952
PA Beta	LaSalle University, Philadelphia	19 May 1953
VA Alpha	Virginia State University, Petersburg	29 Jan 1955
IN Gamma	Anderson University, Anderson	5 April 1957
CA Gamma	California Polytechnic State University, San Luis Obispo	23 May 1958
TN Beta	East Tennessee State University, Johnson City	22 May 1959
PA Gamma	Waynesburg College, Waynesburg	23 May 1959
VA Beta	Radford University, Radford	12 Nov 1959
NE Beta	University of Nebraska—Kearney, Kearney	11 Dec 1959
IN Delta	University of Evansville, Evansville	27 May 1960
OH Epsilon	Marietta College, Marietta	29 Oct 1960

MO Zeta	University of Missouri—Rolla, Rolla	19 May 1961
NE Gamma	Chadron State College, Chadron	19 May 1962
MD Alpha	College of Notre Dame of Maryland, Baltimore	22 May 1963
CA Delta	California State Polytechnic University, Pomona	5 Nov 1964
PA Delta	Marywood University, Scranton	8 Nov 1964
PA Epsilon	Kutztown University of Pennsylvania, Kutztown	3 April 1965
AL Epsilon	Huntingdon College, Montgomery	15 April 1965
PA Zeta	Indiana University of Pennsylvania, Indiana	6 May 1965
AR Alpha	Arkansas State University, State University	21 May 1965
TN Gamma	Union University, Jackson	24 May 1965
WI Beta	University of Wisconsin—River Falls, River Falls	25 May 1965
IA Gamma	Morningside College, Sioux City	25 May 1965
MD Beta	McDaniel College, Westminster	30 May 1965
IL Zeta	Domincan University, River Forest	26 Feb 1967
SC Beta	South Carolina State College, Orangeburg	6 May 1967
PA Eta	Grove City College, Grove City	13 May 1967
NY Eta	Niagara University, Niagara University	18 May 1968
MA Alpha	Assumption College, Worcester	19 Nov 1968
MO Eta	Truman State University, Kirksville	7 Dec 1968
IL Eta	Western Illinois University, Macomb	9 May 1969
OH Zeta	Muskingum College, New Concord	17 May 1969
PA Theta	Susquehanna University, Selinsgrove	26 May 1969
PA Iota	Shippensburg University of Pennsylvania, Shippensburg	1 Nov 1969
MS Delta	William Carey College, Hattiesburg	17 Dec 1970
MO Theta	Evangel University, Springfield	12 Jan 1971
PA Kappa	Holy Family College, Philadelphia	23 Jan 1971
CO Beta	Colorado School of Mines, Golden	4 March 1971
KY Alpha	Eastern Kentucky University, Richmond	27 March 1971
TN Delta	Carson-Newman College, Jefferson City	15 May 1971
NY Iota	Wagner College, Staten Island	19 May 1971
SC Gamma	Winthrop University, Rock Hill	3 Nov 1972
IA Delta	Wartburg College, Waverly	6 April 1973
PA Lambda	Bloomsburg University of Pennsylvania, Bloomsburg	17 Oct 1973
OK Gamma	Southwestern Oklahoma State University, Weatherford	1 May 1973
NY Kappa	Pace University, New York	24 April 1974
TX Eta	Hardin-Simmons University, Abilene	3 May 1975
MO Iota	Missouri Southern State University, Joplin	8 May 1975
GA Alpha	State University of West Georgia, Carrollton	21 May 1975
WV Alpha	Bethany College, Bethany	21 May 1975
FL Beta	Florida Southern College, Lakeland	31 Oct 1976
WI Gamma	University of Wisconsin—Eau Claire, Eau Claire	4 Feb 1978
MD Delta	Frostburg State University, Frostburg	17 Sept 1978
IL Theta	Benedictine University, Lisle	18 May 1979
PA Mu	St. Francis University, Loretto	14 Sept 1979
AL Zeta	Birmingham-Southern College, Birmingham	18 Feb 1981
CT Beta	Eastern Connecticut State University, Willimantic	2 May 1981
NY Lambda	C.W. Post Campus of Long Island University, Brookville	2 May 1983
MO Kappa	Drury University, Springfield	30 Nov 1984
CO Gamma	Fort Lewis College, Durango	29 March 1985
NE Delta	Nebraska Wesleyan University, Lincoln	18 April 1986

TX Iota	McMurry University, Abilene	25 April 1987
PA Nu	Ursinus College, Collegeville	28 April 1987
VA Gamma	Liberty University, Lynchburg	30 April 1987
NY Mu	St. Thomas Aquinas College, Sparkill	14 May 1987
OH Eta	Ohio Northern University, Ada	15 Dec 1987
OK Delta	Oral Roberts University, Tulsa	10 April 1990
CO Delta	Mesa State College, Grand Junction	27 April 1990
PA Xi	Cedar Crest College, Allentown	30 Oct 1990
MO Lambda	Missouri Western State College, St. Joseph	10 Feb 1991
TX Kappa	University of Mary Hardin-Baylor, Belton	21 Feb 1991
SC Delta	Erskine College, Due West	28 April 1991
SD Alpha	Northern State University, Aberdeen	3 May 1992
NY Nu	Hartwick College, Oneonta	14 May 1992
NH Alpha	Keene State College, Keene	16 Feb 1993
LA Gamma	Northwestern State University, Natchitoches	24 March 1993
KY Beta	Cumberland College, Williamsburg	3 May 1993
MS Epsilon	Delta State University, Cleveland	19 Nov 1994
PA Omicron	University of Pittsburgh at Johnstown, Johnstown	10 April 1997
MI Delta	Hillsdale College, Hillsdale	30 April 1997
MI Epsilon	Kettering University, Flint	28 March 1998
KS Zeta	Southwestern College, Winfield	14 April 1998
TN Epsilon	Bethel College, McKenzie	16 April 1998
MO Mu	Harris-Stowe College, St. Louis	25 April 1998
GA Beta	Georgia College and State University, Milledgeville	25 April 1998
AL Eta	University of West Alabama, Livingston	4 May 1998
NY Xi	Buffalo State College, Buffalo	12 May 1998
NC Delta	High Point University, High Point	24 March 1999
PA Pi	Slippery Rock University, Slippery Rock	19 April 1999
TX Lambda	Trinity University, San Antonio	22 November 1999
GA Gamma	Piedmont College, Demorest	7 April 2000
LA Delta	University of Louisiana, Monroe	11 February 2001
GA Delta	Berry College, Mount Berry	21 April 2001
TX Mu	Schreiner University, Kerrville	28 April 2001
NJ Gamma	Monmouth University	21 April 2002
CA Epsilon	California Baptist University, Riverside	21 April 2003
PA Rho	Thiel College, Greenville	13 February 2004
VA Delta	Marymount University, Arlington	26 March 2004
NY Omicron	St. Joseph's College, Patchogue	1 May 2004
IL Iota	Lewis University, Romeoville	26 February 2005
WV Beta	Wheeling Jesuit University, Wheeling	11 March 2005
SC Epsilon	Francis Marion University, Florence	18 March 2005
PA Sigma	Lycoming College, Williamsport	1 April 2005
MO Nu	Columbia College, Columbia	29 April 2005
MD Epsilon	Villa Julie College, Stevenson	3 December 2005
NJ Delta	Centenary College, Hackettstown	1 December 2006
NY Pi	Mount Saint Mary College, Newburgh	20 March 2007